6 Kaon mixing

The mixing of neutral pseudoscalar mesons plays an important role in the understanding of the physics of CP violation. In this section we discuss $K^0-\bar{K}^0$ oscillations, which probe the physics of indirect CP violation. Extensive reviews on the subject can be found in Refs. [1–3]. For the most part we shall focus on kaon mixing in the SM. The case of Beyond-the-Standard-Model (BSM) contributions is discussed in section 6.3.

6.1 Indirect CP violation and ϵ_K in the SM

Indirect CP violation arises in $K_L\to\pi\pi$ transitions through the decay of the CP $=+1$ component of K_L into two pions (which are also in a CP $=+1$ state). Its measure is defined as

$$\epsilon_K \;=\; \frac{\mathcal{A}[K_L\to(\pi\pi)_{I=0}]}{\mathcal{A}[K_S\to(\pi\pi)_{I=0}]}\;, \tag{99}$$

with the final state having total isospin zero. The parameter ϵ_K may also be expressed in terms of $K^0-\bar{K}^0$ oscillations. In particular, to lowest order in the electroweak theory, the contribution to these oscillations arises from so-called box diagrams, in which two W bosons and two "up-type" quarks (i.e. up, charm, top) are exchanged between the constituent down and strange quarks of the K mesons. The loop integration of the box diagrams can be performed exactly. In the limit of vanishing external momenta and external quark masses, the result can be identified with an effective four-fermion interaction, expressed in terms of the "effective Hamiltonian"

$$\mathcal{H}_{\rm eff}^{\Delta S=2} \;=\; \frac{G_F^2 M_{\rm W}^2}{16\pi^2}\mathcal{F}^0 Q^{\Delta S=2} \;+\; {\rm h.c.}\;. \tag{100}$$

In this expression, G_F is the Fermi coupling, $M_{\rm W}$ the W-boson mass, and

$$Q^{\Delta S=2} = [\bar{s}\gamma_\mu(1-\gamma_5)d]\,[\bar{s}\gamma_\mu(1-\gamma_5)d] \equiv O_{\rm VV+AA} - O_{\rm VA+AV}\;, \tag{101}$$

is a dimension-six, four-fermion operator. The function $\mathcal{F}^0$ is given by

$$\mathcal{F}^0 \;=\; \lambda_c^2 S_0(x_c) \;+\; \lambda_t^2 S_0(x_t) \;+\; 2\lambda_c\lambda_t S_0(x_c,x_t)\;, \tag{102}$$

where $\lambda_a = V_{as}^* V_{ad}$, and $a=c\,,t$ denotes a flavour index. The quantities $S_0(x_c)$, $S_0(x_t)$ and $S_0(x_c,x_t)$ with $x_c = m_c^2/M_{\rm W}^2$, $x_t = m_t^2/M_{\rm W}^2$ are the Inami-Lim functions [4], which express the basic electroweak loop contributions without QCD corrections. The contribution of the up quark, which is taken to be massless in this approach, has been taken into account by imposing the unitarity constraint $\lambda_u+\lambda_c+\lambda_t=0$.

When strong interactions are included, $\Delta S=2$ transitions can no longer be discussed at the quark level. Instead, the effective Hamiltonian must be considered between mesonic initial and final states. Since the strong coupling is large at typical hadronic scales, the resulting weak matrix element cannot be calculated in perturbation theory. The operator product expansion (OPE) does, however, factorize long- and short- distance effects. For energy scales below the charm threshold, the $K^0-\bar{K}^0$ transition amplitude of the effective Hamiltonian can be expressed as

$$\begin{aligned}\langle \bar{K}^0|\mathcal{H}_{\rm eff}^{\Delta S=2}|K^0\rangle \;=\; &\frac{G_F^2 M_{\rm W}^2}{16\pi^2}\Big[\lambda_c^2 S_0(x_c)\eta_1 \;+\; \lambda_t^2 S_0(x_t)\eta_2 \;+\; 2\lambda_c\lambda_t S_0(x_c,x_t)\eta_3\Big] \\ \times\left(\frac{\bar{g}(\mu)^2}{4\pi}\right)^{-\gamma_0/(2\beta_0)} &\exp\left\{\int_0^{\bar{g}(\mu)} dg\left(\frac{\gamma(g)}{\beta(g)} \;+\; \frac{\gamma_0}{\beta_0 g}\right)\right\}\langle \bar{K}^0|Q_{\rm R}^{\Delta S=2}(\mu)|K^0\rangle \;+\; {\rm h.c.}\;,\end{aligned} \tag{103}$$

where $\bar{g}(\mu)$ and $Q_{\rm R}^{\Delta S=2}(\mu)$ are the renormalized gauge coupling and four-fermion operator in some renormalization scheme. The factors η_1, η_2 and η_3 depend on the renormalized coupling $\bar{g}$, evaluated at the various flavour thresholds m_t, m_b, m_c and $M_{\rm W}$, as required by the OPE and RG-running procedure that separate high- and low-energy contributions. Explicit expressions can be found in Refs. [2] and references therein, except that η_1 and η_3 have been recently calculated to NNLO in Refs. [5] and [6], respectively. We follow the same conventions for the RG equations as in Ref. [2]. Thus the Callan-Symanzik function and the anomalous dimension $\gamma(\bar{g})$ of $Q^{\Delta S=2}$ are defined by

$$\frac{d\bar{g}}{d\ln\mu} = \beta(\bar{g})\,, \qquad \frac{dQ_{\rm R}^{\Delta S=2}}{d\ln\mu} = -\gamma(\bar{g})\,Q_{\rm R}^{\Delta S=2}\,, \tag{104}$$

with perturbative expansions

$$\begin{aligned}\beta(g) &= -\beta_0\frac{g^3}{(4\pi)^2} - \beta_1\frac{g^5}{(4\pi)^4} - \cdots \\ \gamma(g) &= \gamma_0\frac{g^2}{(4\pi)^2} + \gamma_1\frac{g^4}{(4\pi)^4} + \cdots .\end{aligned} \tag{105}$$

We stress that β_0, β_1 and γ_0 are universal, i.e. scheme independent. $K^0-\bar{K}^0$ mixing is usually considered in the naive dimensional regularization (NDR) scheme of $\overline{\rm MS}$, and below we specify the perturbative coefficient γ_1 in that scheme:

$$\begin{aligned}\beta_0 &= \left\{\frac{11}{3}N - \frac{2}{3}N_f\right\}, \qquad \beta_1 = \left\{\frac{34}{3}N^2 - N_f\left(\frac{13}{3}N - \frac{1}{N}\right)\right\}, \\ \gamma_0 &= \frac{6(N-1)}{N}, \qquad \gamma_1 = \frac{N-1}{2N}\left\{-21 + \frac{57}{N} - \frac{19}{3}N + \frac{4}{3}N_f\right\}.\end{aligned} \tag{106}$$

Note that for QCD the above expressions must be evaluated for $N=3$ colours, while N_f denotes the number of active quark flavours. As already stated, Eq. (103) is valid at scales below the charm threshold, after all heavier flavours have been integrated out, i.e. $N_f = 3$.

In Eq. (103), the terms proportional to η_1, η_2 and η_3, multiplied by the contributions containing $\bar{g}(\mu)^2$, correspond to the Wilson coefficient of the OPE, computed in perturbation theory. Its dependence on the renormalization scheme and scale μ is canceled by that of the weak matrix element $\langle\bar{K}^0|Q_{\rm R}^{\Delta S=2}(\mu)|K^0\rangle$. The latter corresponds to the long-distance effects of the effective Hamiltonian and must be computed nonperturbatively. For historical, as well as technical reasons, it is convenient to express it in terms of the B parameter $B_{\rm K}$, defined as

$$B_{\rm K}(\mu) = \frac{\left\langle\bar{K}^0\left|Q_{\rm R}^{\Delta S=2}(\mu)\right|K^0\right\rangle}{\frac{8}{3}f_{\rm K}^2 m_{\rm K}^2}\,. \tag{107}$$

The four-quark operator $Q^{\Delta S=2}(\mu)$ is renormalized at scale μ in some regularization scheme, for instance, NDR-$\overline{\rm MS}$. Assuming that $B_{\rm K}(\mu)$ and the anomalous dimension $\gamma(g)$ are both known in that scheme, the renormalization group independent (RGI) B parameter $\hat{B}_{\rm K}$ is related to $B_{\rm K}(\mu)$ by the exact formula

$$\hat{B}_{\rm K} = \left(\frac{\bar{g}(\mu)^2}{4\pi}\right)^{-\gamma_0/(2\beta_0)} \exp\left\{\int_0^{\bar{g}(\mu)} dg\left(\frac{\gamma(g)}{\beta(g)} + \frac{\gamma_0}{\beta_0 g}\right)\right\} B_{\rm K}(\mu)\;. \tag{108}$$

At NLO in perturbation theory the above reduces to

$$\hat{B}_{\rm K} = \left(\frac{\bar{g}(\mu)^2}{4\pi}\right)^{-\gamma_0/(2\beta_0)} \left\{1 + \frac{\bar{g}(\mu)^2}{(4\pi)^2}\left[\frac{\beta_1\gamma_0 - \beta_0\gamma_1}{2\beta_0^2}\right]\right\} B_{\rm K}(\mu)\ . \tag{109}$$

To this order, this is the scale-independent product of all μ-dependent quantities in Eq. (103).

Lattice QCD calculations provide results for $B_K(\mu)$. These results are, however, usually obtained in intermediate schemes other than the continuum $\overline{\rm MS}$ scheme used to calculate the Wilson coefficients appearing in Eq. (103). Examples of intermediate schemes are the RI/MOM scheme [7] (also dubbed the "Rome-Southampton method") and the Schrödinger functional (SF) scheme [8]. These schemes are used as they allow a nonperturbative renormalization of the four-fermion operator, using an auxiliary lattice simulation. This allows $B_K(\mu)$ to be calculated with percent-level accuracy, as described below.

In order to make contact with phenomenology, however, and in particular to use the results presented above, one must convert from the intermediate scheme to the $\overline{\rm MS}$ scheme or to the RGI quantity $\hat{B}_{\rm K}$. This conversion relies on one or two-loop perturbative matching calculations, the truncation errors in which are, for many recent calculations, the dominant source of error in $\hat{B}_{\rm K}$ (see, for instance, Refs. [9–13]). While this scheme-conversion error is not, strictly speaking, an error of the lattice calculation itself, it must be included in results for the quantities of phenomenological interest, namely $B_K(\overline{\rm MS}, 2\,{\rm GeV})$ and $\hat{B}_{\rm K}$. We note that this error can be minimized by matching between the intermediate scheme and $\overline{\rm MS}$ at as large a scale μ as possible (so that the coupling which determines the rate of convergence is minimized). Recent calculations have pushed the matching μ up to the range $3-3.5\,{\rm GeV}$. This is possible because of the use of nonperturbative RG running determined on the lattice [10, 12, 14]. The Schrödinger functional offers the possibility to run nonperturbatively to scales $\mu \sim M_{\rm W}$ where the truncation error can be safely neglected. However, so far this has been applied only for two flavours of Wilson quarks [15].

Perturbative truncation errors in Eq. (103) also affect the Wilson coefficients η_1, η_2 and η_3. It turns out that the largest uncertainty arises from the charm quark contribution $\eta_1 = 1.87(76)$ [5]. Although it is now calculated at NNLO, the series shows poor convergence. The net effect is that the uncertainty in η_1 is larger than that in present lattice calculations of B_K.

In the Standard Model, ϵ_K receives contributions from: 1) short distance physics given by $\Delta S = 2$ "box diagrams" involving $W^\pm$ bosons and u, c and t quarks; 2) long distance physics from light hadrons contributing to the imaginary part of the dispersive amplitude M_{12} used in the two component description of $K^0 - \bar{K}^0$ mixing; 3) the imaginary part of the absorptive amplitude Γ_{12} from $K^0 - \bar{K}^0$ mixing; and 4) ${\rm Im}(A_0)/{\rm Re}(A_0)$. The terms in this decomposition can vary with phase conventions. It is common to represent contribution 1 by

$${\rm Im}(M_{12}^{\rm SD}) \equiv \frac{1}{2m_K}{\rm Im}[\langle \bar{K}^0|\mathcal{H}_{\rm eff}^{\Delta S=2}|K^0\rangle]^* \tag{110}$$

and contribution 2 by $M_{12}^{\rm LD}$. Contribution 3 can be related to ${\rm Im}(A_0)/{\rm Re}(A_0)$, yielding [3, 16–19]

$$\epsilon_K \;=\; \exp(i\phi_\epsilon)\,\sin(\phi_\epsilon)\left[\frac{{\rm Im}(M_{12}^{\rm SD})}{\Delta M_K} + \frac{{\rm Im}(M_{12}^{\rm LD})}{\Delta M_K} + \frac{{\rm Im}(A_0)}{{\rm Re}(A_0)}\right] \tag{111}$$

for λ_u real and positive; the phase of ϵ_K is given by

$$\phi_\epsilon \;=\; \arctan\frac{\Delta M_K}{\Delta\Gamma_K/2}\ . \tag{112}$$

The quantities ΔM_K and $\Delta\Gamma_K$ are the mass and decay width differences between long- and short-lived neutral kaons, while A_0 is the amplitude of the kaon decay into an isospin-0 two pion state. Experimentally known values of the above quantities are [20]:

$$\begin{aligned} |\epsilon_K| &= 2.228(11) \times 10^{-3} \,, \\ \phi_\epsilon &= 43.52(5)^\circ \,, \\ \Delta M_K &= 3.4839(59) \times 10^{-12}\,\mathrm{MeV} \,, \\ \Delta\Gamma_K &= 7.3382(33) \times 10^{-15}\,\mathrm{GeV} \,. \end{aligned} \tag{113}$$

A recent analytical estimate of the contributions of $M_{12}^{\rm LD}$ (Refs. [18, 19]) leads to

$$\epsilon_K \;=\; \exp(i\phi_\epsilon)\;\sin(\phi_\epsilon)\;\Big[\frac{\mathrm{Im}(M_{12}^{\rm SD})}{\Delta M_K} \;+\; \rho\frac{\mathrm{Im}(A_0)}{\mathrm{Re}(A_0)}\Big] \,. \tag{114}$$

A phenomenological estimate for $\xi = \mathrm{Im}(A_0)/\,\mathrm{Re}(A_0)$ can be determined using the experimental value of ϵ'/ϵ [19]

$$\xi = -6.0(1.5)\cdot 10^{-4}\sqrt{2}|\epsilon_K| = -1.9(5)\cdot 10^{-4}. \tag{115}$$

A more precise result has been obtained from the ratio of amplitudes $\mathrm{Im}(A_2)/\,\mathrm{Re}(A_2)$ computed in lattice QCD [21] (where A_2 denotes the $\Delta I = 3/2$ decay amplitude for $K \to \pi\pi$):

$$\xi = -1.6(2)\cdot 10^{-4}. \tag{116}$$

The value of ξ can then be combined with a χPT-based estimate for the long-range contribution, i.e. $\rho = 0.6(3)$ [19]. Overall, the combination $\rho\xi$ leads to a suppression of $|\epsilon_K|$ by 6(2)% relative to the naive estimate (i.e. the first term in square brackets in Eq. (111)), regardless of whether the phenomenological or lattice estimate for ξ is used. The uncertainty in the suppression factor is dominated by the error on ρ. Although this is a small correction, we note that its contribution to the error of ϵ_K is larger than that arising from the value of $B_{\rm K}$ reported below.

Efforts are under way to compute both the real and imaginary long-distance contribution to the $K_L - K_S$ mass difference in lattice QCD [22–24]. However, the results are not yet precise enough to improve the accuracy in the determination of the parameter ρ.

6.2 Lattice computation of $B_{\rm K}$

Lattice calculations of $B_{\rm K}$ are affected by the same systematic effects discussed in previous sections. However, the issue of renormalization merits special attention. The reason is that the multiplicative renormalizability of the relevant operator $Q^{\Delta S=2}$ is lost once the regularized QCD action ceases to be invariant under chiral transformations. For Wilson fermions, $Q^{\Delta S=2}$ mixes with four additional dimension-six operators, which belong to different representations of the chiral group, with mixing coefficients that are finite functions of the gauge coupling. This complicated renormalization pattern was identified as the main source of systematic error in earlier, mostly quenched calculations of $B_{\rm K}$ with Wilson quarks. It can be bypassed via the implementation of specifically designed methods, which are either based on Ward identities [25] or on a modification of the Wilson quark action, known as twisted mass QCD [26, 27].

An advantage of staggered fermions is the presence of a remnant $U(1)$ chiral symmetry. However, at nonvanishing lattice spacing, the symmetry among the extra unphysical degrees of freedom (tastes) is broken. As a result, mixing with other dimension-six operators cannot be avoided in the staggered formulation, which complicates the determination of the B parameter. The effects of the broken taste symmetry are usually treated via an effective field theory, such as staggered Chiral Perturbation Theory (SχPT).

Fermionic lattice actions based on the Ginsparg-Wilson relation [28] are invariant under the chiral group, and hence four-quark operators such as $Q^{\Delta S=2}$ renormalize multiplicatively. However, depending on the particular formulation of Ginsparg-Wilson fermions, residual chiral symmetry breaking effects may be present in actual calculations. For instance, in the case of domain wall fermions, the finiteness of the extra 5th dimension implies that the decoupling of modes with different chirality is not exact, which produces a residual nonzero quark mass in the chiral limit. Whether or not a significant mixing with dimension-six operators is induced as well must be investigated on a case-by-case basis.

Recent lattice QCD calculations of B_K have been performed with $N_f = 2+1+1$ dynamical quarks [29], and we want to mention a few conceptual issues that arise in this context. As described in section 6.1, kaon mixing is expressed in terms of an effective four-quark interaction $Q^{\Delta S=2}$, considered below the charm threshold. When the matrix element of $Q^{\Delta S=2}$ is evaluated in a theory that contains a dynamical charm quark, the resulting estimate for B_K must then be matched to the three-flavour theory which underlies the effective four-quark interaction.[1] In general, the matching of $2+1$-flavour QCD with the theory containing $2+1+1$ flavours of sea quarks below the charm threshold can be accomplished by adjusting the coupling and quark masses of the $N_f = 2+1$ theory so that the two theories match at energies $E < m_c$. The corrections associated with this matching are of order $(E/m_c)^2$, since the subleading operators have dimension eight [30]. When the kaon mixing amplitude is considered, the matching also involves the relation between the relevant box graphs and the effective four-quark operator. In this case, corrections of order $(E/m_c)^2$ arise not only from the charm quarks in the sea, but also from the valence sector, since the charm quark propagates in the box diagrams. One expects that the sea quark effects are subdominant, as they are suppressed by powers of α_s. We note that the original derivation of the effective four-quark interaction is valid up to corrections of order $(E/m_c)^2$. While the kaon mixing amplitudes evaluated in the $N_f = 2+1$ and $2+1+1$ theories are thus subject to corrections of the same order in E/m_c as the derivation of the conventional four-quark interaction, the general conceptual issue regarding the calculation of B_K in QCD with $N_f = 2+1+1$ flavours should be addressed in detail in future calculations.

Another issue in this context is how the lattice scale and the physical values of the quark masses are determined in the $2+1$ and $2+1+1$ flavour theories. Here it is important to consider in which way the quantities used to fix the bare parameters are affected by a dynamical charm quark. Apart from a brief discussion in Ref. [29], these issues have not been fully worked out in the literature, but these kinds of mismatches were seen in simple lattice-QCD observables as quenched calculations gave way to $N_f = 2$ and then $2+1$ flavour results. Given the scale of the charm quark mass relative to the scale of B_K, we expect these errors to be modest, but a more quantitative understanding is needed as statistical errors on B_K are reduced. Within this review we will not discuss this issue further.

Below we focus on recent results for $B_{\rm K}$, obtained for $N_f = 2, 2+1$ and $2+1+1$ flavours

[1] We thank Martin Lüscher for an interesting discussion on this issue.

of dynamical quarks. A compilation of results is shown in Tabs. 25 and 26, as well as Fig. 15. An overview of the quality of systematic error studies is represented by the colour coded entries in Tabs. 25 and 26. In Appendix B.5 we gather the simulation details and results from different collaborations, the values of the most relevant lattice parameters, and comparative tables on the various estimates of systematic errors.

Some of the groups whose results are listed in Tabs. 25 and 26 do not quote results for both $B_{\rm K}(\overline{\rm MS}, 2\,{\rm GeV})$ – which we denote by the shorthand $B_{\rm K}$ from now on – and $\hat{B}_{\rm K}$. This concerns Refs. [31–33] for $N_f = 2$, Refs.[9, 10, 12, 13] for $2+1$ and Ref. [29] for $2+1+1$ flavours. In these cases we perform the conversion ourselves by evaluating the proportionality factor in Eq. (109) at $\mu = 2\,{\rm GeV}$, using the following procedure: For $N_f = 2+1$ we use the value $\alpha_s(M_{\rm Z}) = 0.1185$ from the 2014 edition of the PDG [20] and run it across the quark thresholds at $m_b = 4.18\,{\rm GeV}$ and $m_c = 1.275\,{\rm GeV}$, and then run up in the three-flavour theory to $\mu = 2\,{\rm GeV}$. All running is done using the four-loop RG β-function. The resulting value of $\alpha_s^{\overline{\rm MS}}(2\,{\rm GeV}) = 0.2967$ is then used to evaluate $\hat{B}_{\rm K}/B_{\rm K}$ in perturbation theory at NLO, which gives $\hat{B}_{\rm K}/B_{\rm K} = 1.369$ in the three-flavour theory. This value of the conversion factor has also been applied to the result computed in QCD with $N_f = 2+1+1$ flavours of dynamical quarks [29].

In two-flavour QCD one can insert the updated nonperturbative estimate for the Λ parameter by the ALPHA Collaboration [34], i.e. $\Lambda^{(2)} = 310(20)\,{\rm MeV}$, into the NLO expressions for α_s. The resulting value of the perturbative conversion factor $\hat{B}_K/B_K$ for $N_f = 2$ is then equal to 1.386. However, since the running coupling in the $\overline{\rm MS}$ scheme enters at several stages in the entire matching and running procedure, it is difficult to use this estimate of α_s consistently without a partial reanalysis of the data in Refs. [31–33]. We have therefore chosen to apply the conversion factor of 1.369 not only to results obtained for $N_f = 2+1$ flavours but also to the two-flavour theory (in cases where only one of $\hat{B}_K$ and B_K are quoted). We note that the difference between 1.386 and 1.369 will produce an ambiguity of the order of 1%, which is well below the overall uncertainties in Refs. [31, 32]. We have indicated explicitly in Tab. 26 in which way the conversion factor 1.369 has been applied to the results of Refs. [31–33].

Recent results for the kaon B parameter have been reported for $N_f = 2+1+1$ (ETM 15 [29]), $N_f = 2{+}1$ (RBC/UKQCD 14B [12], RBC/UKQCD 16 [35], SWME 13A [36], SWME 14 [11], SWME 15A [13]) and $N_f = 2$ (ETM 12D [33]). We briefly discuss the main features of these calculations below.

The calculation by ETM 15 [29] employs Osterwalder-Seiler valence quarks on twisted-mass dynamical quark ensembles. Both valence and sea quarks are tuned to maximal twist. This mixed action setup guarantees that the four-fermion matrix elements are automatically $\mathcal{O}(a)$ improved and free of wrong chirality mixing effects. The calculation has been carried out at three values of the lattice spacing ($a \simeq 0.06 - 0.09$ fm). Light pseudoscalar mass values are in the range $210 - 450\,{\rm MeV}$. The spatial lattice sizes vary between 2.1 to 2.9 fm and correspond to $M_{\pi,\rm min}L \simeq 3.2{-}3.5$. Finite volume effects are investigated at the coarsest lattice spacing by controlling the consistency of results obtained at two lattice volumes at 280 MeV for the light pseudoscalar mass. The determination of the bag parameter is performed using simultaneous chiral and continuum fits. The renormalization factors have been evaluated using the RI/MOM technique for $N_f = 4$ degenerate Wilson twisted-mass dynamical quark gauge configurations generated for this purpose. In order to gain control over discretization effects the evaluation of the renormalization factors has been carried out following two different methods. The uncertainty from the RI computation is estimated at 2%. The conversion to

$\overline{\mathrm{MS}}$ produces an additional 0.6% of systematic error. The overall uncertainty for the bag parameter is computed from a distribution of several results, each one of them corresponding to a variant of the analysis procedure.

The collection of results from the SWME collaboration [11, 13, 36–38, 40] have all been obtained using a mixed action, i.e. HYP-smeared valence staggered quarks on the Asqtad improved, rooted staggered MILC ensembles. For the latest set of results, labelled SWME 14, 15A [11, 13] an extended set of ensembles, comprising finer lattice spacings and a smallest pion mass of 174 MeV has been added to the calculation. The final estimate for B_K is obtained from a combined chiral and continuum extrapolation using the data computed for the three finest lattice spacings. The dominant systematic error of 4.4% is associated with the matching factor between the lattice and $\overline{\mathrm{MS}}$ schemes. It has been computed in perturbation theory at one loop, and its error was estimated assuming a missing two-loop matching term of size $1 \times \alpha(1/a)^2$, i.e. with no factors of $1/(4\pi)$ included. Different functional forms for the chiral fits contribute another 2% to the error budget. It should also be noted that Bayesian priors are used to constrain some of the coefficients in the chiral ansatz. The total systematic error amounts to about 5%. Compared to the earlier calculations of SWME one finds that "the overall error is only slightly reduced, but, more importantly, the methods of estimating errors have been improved" [11].

The RBC and UKQCD Collaborations have updated their value for B_K using $N_f = 2+1$ flavours of domain wall fermions [12]. Previous results came from ensembles at three different lattice spacings with unitary pion masses in the range of 170 to 430 MeV. The new work adds an ensemble with essentially physical light and strange quark masses at two of the lattice spacings, along with a third finer lattice with 370 MeV pion masses. This finer ensemble provides an additional constraint on continuum extrapolations. Lattice spacings and quark masses are determined via a combined continuum and chiral extrapolation to all ensembles. With lattice spacings at hand, nonperturbative renormalization and nonperturbative step scaling are used to find the renormalized value of B_K at 3 GeV in the RI-SMOM(γ^μ, γ^μ) and RI-SMOM$(\not{q}, \not{q})$ schemes for all of the ensembles. These B_K values for each pion mass are determined for the physical strange quark mass through valence strange quark interpolations/extrapolations and dynamical strange quark mass reweighting. The light quark mass dependence is then fit to $SU(2)$ chiral perturbation theory. Because the new ensembles have quark masses within a few percent of their physical values, the systematic error related to the extrapolation to physical values is neglected. The new physical point ensembles have $(5.5\,\mathrm{fm})^3$ volumes, and chiral perturbation theory fits with and without finite volume corrections differ by 10-20% of the statistical errors, so no finite volume error is quoted. The fits are dominated by the physical point ensembles, which have small errors. Fits with B_K normalized in both RI-SMOM schemes are done, and the difference is used to estimate the systematic error due to nonperturbative renormalization. In another recent paper [35], RBC/UKQCD reported a determination of B_K obtained as part of their study of kaon mixing in extensions of the SM. While the procedure to determine B_K are very similar to RBC/UKQCD 14B, the calculation in RBC/UKQCD 16 [35] is based only on a subset of the ensembles studied in [12]. Therefore, the result for B_K reported in [35] can neither be considered an update of RBC/UKQCD 14B, nor an independent new result.

The $N_f = 2$ calculation described in ETM 12D [33] uses a mixed action setup employing twisted-mass dynamical quarks and Osterwalder-Seiler quarks in the valence, both tuned to maximal twist. The work of ETM 12D is an update of the calculation of ETM 10A [32]. The main addition is the inclusion of a fourth (superfine) lattice spacing ($a \simeq 0.05$ fm). Thus, the computation is performed at four values of the lattice spacing ($a \simeq 0.05 - 0.1$ fm), and the lightest simulated value of the light pseudoscalar mass is about 280 MeV. Final results are obtained with combined chiral and continuum fits. Finite volume effects are studied at one value of the lattice spacing ($a \simeq 0.08$ fm), and it is found that results obtained on two lattice volumes, namely for $L = 2.2$ and 2.9 fm at $M_\pi \approx 300$ MeV are in good agreement within errors. The four- and two-fermion renormalization factors needed in the bag parameter evaluation are computed nonperturbatively using the Rome-Southampton method. The systematic error due to the matching of RI and $\overline{\mathrm{MS}}$ schemes is estimated to be 2.5%.

We now describe our procedure for obtaining global averages. The rules of section 2.1 stipulate that results free of red tags and published in a refereed journal may enter an average. Papers that at the time of writing are still unpublished but are obvious updates of earlier published results can also be taken into account.

There is only one result for $N_f = 2+1+1$, computed by the ETM Collaboration [29]. Since it is free of red tags, it qualifies as the currently best global estimate, i.e.

$$N_f = 2+1+1: \quad \hat{B}_{\mathrm{K}} = 0.717(18)(16)\,, \quad B_{\mathrm{K}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.524(13)(12) \quad \text{Ref. [29]}. \tag{117}$$

The bulk of results for the kaon B parameter has been obtained for $N_f = 2+1$. As in the previous edition of the FLAG review [48] we include the results from SWME [11, 13, 36], despite the fact that nonperturbative information on the renormalization factors is not available. Instead, the matching factor has been determined in perturbation theory at one loop, but with a sufficiently conservative error of 4.4%. As described above, the result in RBC/UKQCD 16 [35] cannot be considered an update of the earlier estimate in RBC/UKQCD 14B, and hence it is not included in the FLAG average.

Thus, for $N_f = 2+1$ our global average is based on the results of BMW 11 [14], Laiho 11 [9], RBC/UKQCD 14B [12] and SWME 15A [13]. The last three are the latest updates from a series of calculations by the same collaborations. Our procedure is as follows: in a first step statistical and systematic errors of each individual result for the RGI B parameter, $\hat{B}_{\mathrm{K}}$, are combined in quadrature. Next, a weighted average is computed from the set of results. For the final error estimate we take correlations between different collaborations into account. To this end we note that we consider the statistical and finite-volume errors of SWME 15A and Laiho 11 to be correlated, since both groups use the Asqtad ensembles generated by the MILC Collaboration. Laiho 11 and RBC/UKQCD 14B both use domain wall quarks in the valence sector and also employ similar procedures for the nonperturbative determination of matching factors. Hence, we treat the quoted renormalization and matching uncertainties by the two groups as correlated. After constructing the global covariance matrix according to Schmelling [49], we arrive at

$$N_f = 2+1: \qquad \hat{B}_{\mathrm{K}} = 0.7625(97) \qquad \text{Refs. [9, 12–14]}, \tag{118}$$

with $\chi^2/\mathrm{d.o.f.} = 0.675$. After applying the NLO conversion factor $\hat{B}_{\mathrm{K}}/B_{\mathrm{K}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 1.369$, this translates into

$$N_f = 2+1: \qquad B_{\mathrm{K}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.5570(71) \qquad \text{Refs. [9, 12–14]}. \tag{119}$$

These values and their uncertainties are very close to the global estimates quoted in the previous edition of the FLAG review [48]. Note, however, that the statistical errors of each calculation entering the global average have now been reduced to a level that makes them

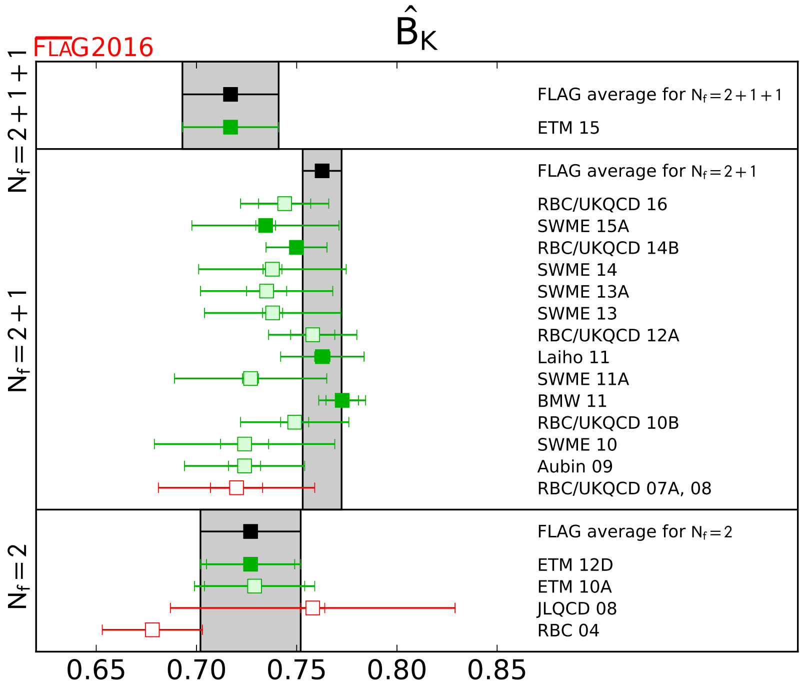

Figure 15: Recent unquenched lattice results for the RGI B parameter $\hat{B}_{\rm K}$. The grey bands indicate our global averages described in the text. For $N_f = 2+1+1$ and $N_f = 2$ the global estimate coincide with the results by ETM 12D and ETM 10A, respectively.

statistically incompatible. It is only because of the relatively large systematic errors that the weighted average produces a value of $\mathcal{O}(1)$ for the reduced χ^2.

Passing over to describing the results computed for $N_f = 2$ flavours, we note that there is only the set of results published in ETM 12D [33] and ETM 10A [32] that allow for an extensive investigation of systematic uncertainties. We identify the result from ETM 12D [33], which is an update of ETM 10A, with the currently best global estimate for two-flavour QCD, i.e.

$$N_f = 2: \quad \hat{B}_{\rm K} = 0.727(22)(12), \quad B_{\rm K}^{\overline{\rm MS}}(2\,{\rm GeV}) = 0.531(16)(19) \quad \text{Ref. [33]}. \tag{120}$$

The result in the $\overline{\rm MS}$ scheme has been obtained by applying the same conversion factor of 1.369 as in the three-flavour theory.

6.3 Kaon BSM B parameters

We now report on lattice results concerning the matrix elements of operators that encode the effects of physics beyond the Standard Model (BSM) to the mixing of neutral kaons. In this theoretical framework both the SM and BSM contributions add up to reproduce the experimentally observed value of ϵ_K. Since BSM contributions involve heavy but unobserved particles they are short-distance dominated. The effective Hamiltonian for generic $\Delta S = 2$ processes including BSM contributions reads

$$\mathcal{H}^{\Delta S=2}_{\rm eff,BSM} = \sum_{i=1}^{5} C_i(\mu) Q_i(\mu), \tag{121}$$

where Q_1 is the four-quark operator of Eq. (101) that gives rise to the SM contribution to ϵ_K. In the so-called SUSY basis introduced by Gabbiani et al. [50] the (parity-even) operators $Q_2, \ldots, Q_5$ read [2]

$$\begin{aligned}
Q_2 &= \left(\bar{s}^a(1-\gamma_5)d^a\right)\left(\bar{s}^b(1-\gamma_5)d^b\right),\\
Q_3 &= \left(\bar{s}^a(1-\gamma_5)d^b\right)\left(\bar{s}^b(1-\gamma_5)d^a\right),\\
Q_4 &= \left(\bar{s}^a(1-\gamma_5)d^a\right)\left(\bar{s}^b(1+\gamma_5)d^b\right),\\
Q_5 &= \left(\bar{s}^a(1-\gamma_5)d^b\right)\left(\bar{s}^b(1+\gamma_5)d^a\right),
\end{aligned} \tag{122}$$

where a and b denote colour indices. In analogy to the case of $B_{\rm K}$ one then defines the B parameters of $Q_2, \ldots, Q_5$ according to

$$B_i(\mu) = \frac{\left\langle \bar{K}^0 \left| Q_i(\mu) \right| K^0 \right\rangle}{N_i \left\langle \bar{K}^0 \left| \bar{s}\gamma_5 d \right| 0 \right\rangle \left\langle 0 \left| \bar{s}\gamma_5 d \right| K^0 \right\rangle}, \quad i = 2, \ldots, 5. \tag{123}$$

The factors $\{N_2, \ldots, N_5\}$ are given by $\{-5/3, 1/3, 2, 2/3\}$, and it is understood that $B_i(\mu)$ is specified in some renormalization scheme, such as $\overline{\rm MS}$ or a variant of the regularization-independent momentum subtraction (RI-MOM) scheme.

The SUSY basis has been adopted in Refs. [29, 33, 35, 51]. Alternatively, one can employ the chiral basis of Buras, Misiak and Urban [52]. The SWME Collaboration prefers the latter, since the anomalous dimension which enters the RG running has been calculated to two loops in perturbation theory [52]. Results obtained in the chiral basis can be easily converted to the SUSY basis via

$$B_3^{\rm SUSY} = \tfrac{1}{2}\left(5B_2^{\rm chiral} - 3B_3^{\rm chiral}\right). \tag{124}$$

The remaining B parameters are the same in both bases. In the following we adopt the SUSY basis and drop the superscript.

Older quenched results for the BSM B parameters can be found in Refs. [53–55]. Recent estimates for $B_2, \ldots, B_5$ have been reported for QCD with $N_f = 2$ (ETM 12D [33]), $N_f = 2+1$ (RBC/UKQCD 12E [51], SWME 13A [36], SWME 14C [56], SWME 15A [13], RBC/UKQCD 16 [35]) and $N_f = 2+1+1$ (ETM 15 [29]) flavours of dynamical quarks. They are listed and compared in Tab. 27 and Fig. 16. In general one finds that the BSM B parameters computed by different collaborations do not show the same level of consistency as the SM kaon mixing parameter B_K discussed previously. The main features of the calculations reported in the table and figure are identical to the case of $B_{\rm K}$ discussed above. We note, in particular, that SWME perform the matching between rooted staggered quarks and the $\overline{\rm MS}$ scheme using perturbation theory at one loop, while RBC/UKQCD and ETMC employ nonperturbative renormalization for domain wall and twisted-mass Wilson quarks, respectively. Control over systematic uncertainties (chiral and continuum extrapolations, finite-volume effects) in $B_2, \ldots, B_5$ is expected to be at the same level as for $B_{\rm K}$, as far as the results by ETM 12D, ETM 15 and SWME 15A are concerned. The calculation by RBC/UKQCD 12E has been performed at a single value of the lattice spacing and a minimum pion mass of 290 MeV. Thus, the results do not benefit from the same improvements regarding control over the chiral and continuum extrapolations as in the case of $B_{\rm K}$ [12].

[2] Thanks to QCD parity invariance we can ignore three more dimension-six operators whose parity conserving parts coincide with the corresponding parity conserving contributions of the operators Q_1, Q_2 and Q_3.

The RBC/UKQCD collaboration have recently extended their calculation of BSM B parameters [35] for $N_f = 2+1$, by considering two values of the lattice spacing, $a \simeq 0.11$ and 0.08 fm, employing ensembles generated using the Iwasaki gauge action and the Shamir domain wall fermionic action. The lattice volumes in the RBC/UKQCD 16 calculation are $24^3 \times 64 \times 16$ for the coarse and $32^3 \times 64 \times 16$ for the fine lattice spacing, while the lowest simulated values for the pseudoscalar mass are about 340 MeV and 300 MeV, respectively. As in the related calculation of B_K (RBC/UKQCD 14B [12]) the renormalization of four-quark operators was performed nonperturbatively in two RI-SMOM schemes, namely $(\not{q}, \not{q})$ and (γ_μ, γ_μ), where the latter was used for the final estimates of $B_2, \ldots, B_5$ quoted in ref. [35]. By comparing the results obtained in the conventional RI-MOM and the two RI-SMOM schemes, RBC/UKQCD 16 report significant discrepancies for B_4 and B_5 in the $\overline{\mathrm{MS}}$ scheme at the scale of 3 GeV, which amount up to 2.8σ in the case of B_5. By contrast, the agreement for B_2 and B_3 determined for different intermediate scheme is much better. Based on these findings they claim that these discrepancies are due to uncontrolled systematics coming from the Goldstone boson pole subtraction procedure that is needed in the RI-MOM scheme, while pole subtraction effects are much suppressed in RI-SMOM thanks to the fact that the latter is based on non-exceptional momenta.

The findings by RBC/UKQCD 16 [35] provides evidence that the nonperturbative determination of the matching factors depends strongly on the details in the implementation of the Rome-Southampton method. The use of nonexceptional momentum configurations in the calculation of the vertex functions produces a significant modification of the renormalization factors which affects the matching between $\overline{\mathrm{MS}}$ and the intermediate momentum subtraction scheme. This effect is most pronounced in B_4 and B_5. As a result, the estimates for B_4 and B_5 from RBC/UKQCD 16 are much closer to those of SWME 15A. At the same time, the results for B_2 and B_3 obtained by ETM 15, SWME 15A and RBC/UKQCD 16 are in good agreement within errors.

A detailed look at the most recent calculations reported in ETM 15 [29], SWME 15A [13] and RBC/UKQCD 16 [35] reveals that cutoff effects appear to be larger for the BSM B parameters compared to B_K. Depending on the details of the renormalization procedure and/or the fit *ansatz* for the combined chiral and continuum extrapolation, the results obtained at the coarsest lattice spacing differ by 15–30%. At the same time the available range of lattice spacings is typically much reduced compared to the corresponding calculations of B_K, as can be seen by comparing the quality criteria in Tables 25 and 27. Hence, the impact of the renormalization procedure and the continuum limit on the BSM B parameters certainly requires further investigation.

Finally we present our estimates for the BSM B parameters, quoted in the $\overline{\mathrm{MS}}$-scheme at scale 3 GeV. For $N_f = 2+1$ our estimate is given by the average between the results from SWME 15A and RBC/UKQCD 16, i.e.

$$\begin{aligned}
&N_f = 2+1: && (125)\\
&B_2 = 0.502(14), \quad B_3 = 0.766(32), \quad B_4 = 0.926(19), \quad B_5 = 0.720(38), && \text{Refs. [13, 35].}
\end{aligned}$$

For $N_f = 2{+}1{+}1$ and $N_f = 2$, our estimates coincide with the ones by ETM 15 and ETM 12D, respectively, since there is only one computation for each case. Thus we quote

$$N_f = 2+1+1: \qquad (126)$$
$$B_2 = 0.46(1)(3), \quad B_3 = 0.79(2)(4), \quad B_4 = 0.78(2)(4), \quad B_5 = 0.49(3)(3), \quad \text{Ref. [29]},$$

$$N_f = 2: \qquad (127)$$
$$B_2 = 0.47(2)(1), \quad B_3 = 0.78(4)(2), \quad B_4 = 0.76(2)(2), \quad B_5 = 0.58(2)(2), \quad \text{Ref. [33]}.$$

Based on the above discussion on the effects of employing different intermediate momentum subtraction schemes in the nonperturbative renormalization of the operators, the discrepancy for B_4 and B_5 results between $N_f = 2, 2+1+1$ and $N_f = 2+1$ computations should not be considered an effect associated with the number of dynamical flavours.

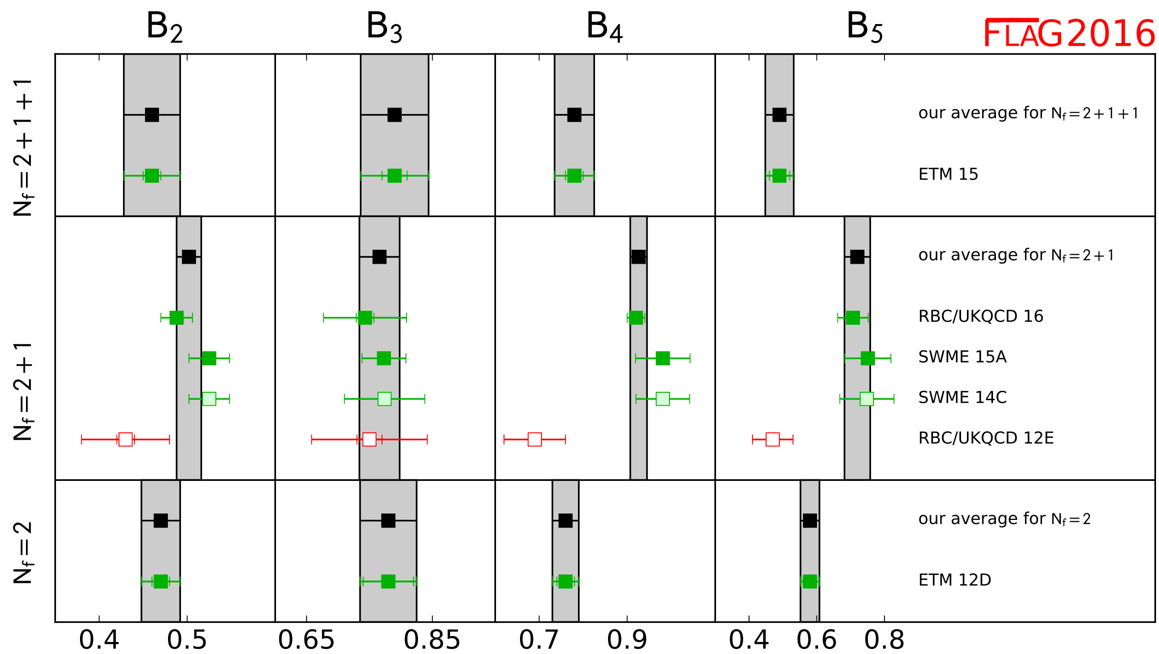

Figure 16: Lattice results for the BSM B parameters defined in the $\overline{\mathrm{MS}}$ scheme at a reference scale of 3 GeV, see Tab. 27.

Collaboration	Ref.	N_f	publication status	continuum extrapolation	chiral extrapolation	finite volume	renormalization	running	$B_K(\overline{\mathrm{MS}}, 2\,\mathrm{GeV})$	$\hat{B}_K$
ETM 15	[29]	2+1+1	A	★	○	○	★	*a*	0.524(13)(12)	0.717(18)(16)[1]
RBC/UKQCD 16	[35]	2+1	A	○	○	○	★	*b*	0.544(9)(13)[3]	0.744(13)(18)
SWME 15A	[13]	2+1	A	★	○	★	○‡	–	0.537(4)(26)	0.735(5)(36)[2]
RBC/UKQCD 14B	[12]	2+1	A	★	★	○	★	*b*	0.5478(18)(110)[3]	0.7499(24)(150)
SWME 14	[11]	2+1	A	★	○	★	○‡	–	0.5388(34)(266)	0.7379(47)(365)
SWME 13A	[36]	2+1	A	★	○	★	○‡	–	0.537(7)(24)	0.735(10)(33)
SWME 13	[37]	2+1	C	★	○	★	○‡	–	0.539(3)(25)	0.738(5)(34)
RBC/UKQCD 12A	[10]	2+1	A	○	★	○	★	*b*	0.554(8)(14)[3]	0.758(11)(19)
Laiho 11	[9]	2+1	C	★	○	○	★	–	0.5572(28)(150)	0.7628(38)(205)[2]
SWME 11A	[38]	2+1	A	★	○	○	○‡	–	0.531(3)(27)	0.727(4)(38)
BMW 11	[14]	2+1	A	★	★	★	★	*c*	0.5644(59)(58)	0.7727(81)(84)
RBC/UKQCD 10B	[39]	2+1	A	○	○	★	★	*d*	0.549(5)(26)	0.749(7)(26)
SWME 10	[40]	2+1	A	★	○	○	○	–	0.529(9)(32)	0.724(12)(43)
Aubin 09	[41]	2+1	A	○	○	○	★	–	0.527(6)(21)	0.724(8)(29)
RBC/UKQCD 07A, 08	[42, 43]	2+1	A	■	○	★	★	–	0.524(10)(28)	0.720(13)(37)
HPQCD/UKQCD 06	[44]	2+1	A	■	○*	★	■	–	0.618(18)(135)	0.83(18)

‡ The renormalization is performed using perturbation theory at one loop, with a conservative estimate of the uncertainty.

* This result has been obtained with only two "light" sea quark masses.

a B_K is renormalized nonperturbatively at scales $1/a \sim 2.2-3.3\,\mathrm{GeV}$ in the $N_f=4$ RI/MOM scheme using two different lattice momentum scale intervals, the first around $1/a$ while the second around 3.5 GeV. The impact of the two ways to the final result is taken into account in the error budget. Conversion to $\overline{\mathrm{MS}}$ is at one-loop at 3 GeV.

b B_K is renormalized nonperturbatively at a scale of 1.4 GeV in two RI/SMOM schemes for $N_f=3$, and then run to 3 GeV using a nonperturbatively determined step-scaling function. Conversion to $\overline{\mathrm{MS}}$ is at one-loop order at 3 GeV.

c B_K is renormalized and run nonperturbatively to a scale of 3.4 GeV in the RI/MOM scheme. nonperturbative and NLO perturbative running agrees down to scales of 1.8 GeV within statistical uncertainties of about 2%.

d B_K is renormalized nonperturbatively at a scale of 2 GeV in two RI/SMOM schemes for $N_f=3$, and then run to 3 GeV using a nonperturbatively determined step-scaling function. Conversion to $\overline{\mathrm{MS}}$ is at one-loop order at 3 GeV.

1 $B_K(\overline{\mathrm{MS}}, 2\,\mathrm{GeV})$ and $\hat{B}_K$ are related using the conversion factor 1.369 *i.e.* the one obtained with $N_f=2+1$.

2 $\hat{B}_K$ is obtained from the estimate for $B_K(\overline{\mathrm{MS}}, 2\,\mathrm{GeV})$ using the conversion factor 1.369.

3 $B_K(\overline{\mathrm{MS}}, 2\,\mathrm{GeV})$ is obtained from the estimate for $\hat{B}_K$ using the conversion factor 1.369.

Table 25: Results for the Kaon B parameter in QCD with $N_f=2+1+1$ and $N_f=2+1$ dynamical flavours, together with a summary of systematic errors. Any available information about nonperturbative running is indicated in the column "running", with details given at the bottom of the table.

Collaboration	Ref.	N_f	publication status	continuum extrapolation	chiral extrapolation	finite volume	renormalization	running	$B_K(\overline{\mathrm{MS}}, 2\,\mathrm{GeV})$	$\hat{B}_K$
ETM 12D	[33]	2	A	★	○	○	★	e	0.531(16)(9)	0.727(22)(12)^1
ETM 10A	[32]	2	A	★	○	○	★	f	0.533(18)(12)^1	0.729(25)(17)
JLQCD 08	[45]	2	A	■	○	■	★	–	0.537(4)(40)	0.758(6)(71)
RBC 04	[31]	2	A	■	■	■†	★	–	0.495(18)	0.678(25)^1
UKQCD 04	[46]	2	A	■	■	■†	■	–	0.49(13)	0.68(18)

† These results have been obtained at $(M_\pi L)_{\min} > 4$ in a lattice box with a spatial extension $L < 2$ fm.

e B_K is renormalized nonperturbatively at scales $1/a \sim 2 - 3.7\,\mathrm{GeV}$ in the $N_f = 2$ RI/MOM scheme. In this scheme, nonperturbative and NLO perturbative running are shown to agree from 4 GeV down to 2 GeV to better than 3% [32, 47].

f B_K is renormalized nonperturbatively at scales $1/a \sim 2 - 3\,\mathrm{GeV}$ in the $N_f = 2$ RI/MOM scheme. In this scheme, nonperturbative and NLO perturbative running are shown to agree from 4 GeV down to 2 GeV to better than 3% [32, 47].

^{1} $B_K(\overline{\mathrm{MS}}, 2\,\mathrm{GeV})$ and $\hat{B}_K$ are related using the conversion factor 1.369 *i.e.* the one obtained with $N_f = 2 + 1$.

Table 26: Results for the Kaon B parameter in QCD with $N_f = 2$ dynamical flavours, together with a summary of systematic errors. Any available information about nonperturbative running is indicated in the column "running", with details given at the bottom of the table.

Collaboration	Ref.	N_f	publication status	continuum extrapolation	chiral extrapolation	finite volume	renormalization	running	B_2	B_3	B_4	B_5
ETM 15	[29]	2+1+1	A	★	○	○	★	a	0.46(1)(3)	0.79(2)(5)	0.78(2)(4)	0.49(3)(3)
RBC/UKQCD 16	[35]	2+1	A	○	○	○	★	b	0.488(7)(17)	0.743(14)(64)	0.920(12)(16)	0.707(8)(44)
SWME 15A	[13]	2+1	A	★	○	★	○†	–	0.525(1)(23)	0.773(6)(35)	0.981(3)(62)	0.751(7)(68)
SWME 14C	[56]	2+1	C	★	○	★	○†	–	0.525(1)(23)	0.774(6)(64)	0.981(3)(61)	0.748(9)(79)
SWME 13A‡	[36]	2+1	A	★	○	★	○†	–	0.549(3)(28)	0.790(30)	1.033(6)(46)	0.855(6)(43)
RBC/ UKQCD 12E	[51]	2+1	A	■	○	★	★	b	0.43(1)(5)	0.75(2)(9)	0.69(1)(7)	0.47(1)(6)
ETM 12D	[33]	2	A	★	○	○	★	c	0.47(2)(1)	0.78(4)(2)	0.76(2)(2)	0.58(2)(2)

† The renormalization is performed using perturbation theory at one loop, with a conservative estimate of the uncertainty.

a B_i are renormalized nonperturbatively at scales $1/a \sim 2.2 - 3.3\,\mathrm{GeV}$ in the $N_f = 4$ RI/MOM scheme using two different lattice momentum scale intervals, with values around $1/a$ for the first and around 3.5 GeV for the second one. The impact of these two ways to the final result is taken into account in the error budget. Conversion to $\overline{\mathrm{MS}}$ is at one loop at 3 GeV.

b The B parameters are renormalized nonperturbatively at a scale of 3 GeV.

c B_i are renormalized nonperturbatively at scales $1/a \sim 2 - 3.7\,\mathrm{GeV}$ in the $N_f = 2$ RI/MOM scheme using two different lattice momentum scale intervals, with values around $1/a$ for the first and around 3 GeV for the second one.

‡ The computation of B_4 and B_5 has been revised in Refs. [13] and [56].

Table 27: Results for the BSM B parameters $B_2, \ldots, B_5$ in the $\overline{\mathrm{MS}}$ scheme at a reference scale of 3 GeV. Any available information on nonperturbative running is indicated in the column "running", with details given at the bottom of the Tab.

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