6 Kaon mixing

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The mixing of neutral pseudoscalar mesons plays an important role in the understanding of the physics of CP violation. In this section we discuss $K^0-\bar{K}^0$ oscillations, which probe the physics of indirect CP violation. Extensive reviews on the subject can be found in Refs. [1–5]. For the most part, we shall focus on kaon mixing in the SM. The case of Beyond-the-Standard-Model (BSM) contributions is discussed in Sec. 6.3.

6.1 Indirect CP violation and ϵ_K in the SM

Indirect CP violation arises in $K_L \to \pi\pi$ transitions through the decay of the CP $= +1$ component of K_L into two pions (which are also in a CP $= +1$ state). Its measure is defined as

$$\epsilon_K \;=\; \frac{\mathcal{A}[K_L \to (\pi\pi)_{I=0}]}{\mathcal{A}[K_S \to (\pi\pi)_{I=0}]}\,, \tag{127}$$

with the final state having total isospin zero. The parameter ϵ_K may also be expressed in terms of $K^0 - \bar{K}^0$ oscillations. In the Standard Model, ϵ_K receives contributions from: (i) short-distance (SD) physics given by $\Delta S = 2$ “box diagrams” involving $W^\pm$ bosons and u, c and t quarks; (ii) the long-distance (LD) physics from light hadrons contributing to the imaginary part of the dispersive amplitude M_{12} used in the two component description of $K^0 - \bar{K}^0$ mixing; (iii) the imaginary part of the absorptive amplitude Γ_{12} from $K^0 - \bar{K}^0$ mixing; and (iv) $\mathrm{Im}(A_0)/\mathrm{Re}(A_0)$, where A_0 is the $K \to (\pi\pi)_{I=0}$ decay amplitude. The various factors in this decomposition can vary with phase conventions. In terms of the $\Delta S = 2$ effective Hamiltonian, $\mathcal{H}_{\rm eff}^{\Delta S=2}$, it is common to represent contribution (i) by

$$\mathrm{Im}(M_{12}^{\rm SD}) \equiv \frac{1}{2m_K}\mathrm{Im}[\langle \bar{K}^0|\mathcal{H}_{\rm eff}^{\Delta S=2}|K^0\rangle], \tag{128}$$

and contribution (ii) by $\mathrm{Im}\, M_{12}^{\rm LD}$. Contribution (iii) can be related to $\mathrm{Im}(A_0)/\mathrm{Re}(A_0)$ since $(\pi\pi)_{I=0}$ states provide the dominant contribution to absorptive part of the integral in Γ_{12}. Collecting the various pieces yields the following expression for the ϵ_K factor [4, 6–9]

$$\epsilon_K \;=\; \exp(i\phi_\epsilon)\,\sin(\phi_\epsilon)\left[\frac{\mathrm{Im}(M_{12}^{\rm SD})}{\Delta M_K} + \frac{\mathrm{Im}(M_{12}^{\rm LD})}{\Delta M_K} + \frac{\mathrm{Im}(A_0)}{\mathrm{Re}(A_0)}\right], \tag{129}$$

where the phase of ϵ_K is given by

$$\phi_\epsilon \;=\; \arctan\frac{\Delta M_K}{\Delta\Gamma_K/2}\,. \tag{130}$$

The quantities ΔM_K and $\Delta\Gamma_K$ are the mass and decay width differences between long- and short-lived neutral kaons. The experimentally known values of the above quantities read [10]:

$$|\epsilon_K| \;=\; 2.228(11)\times 10^{-3}\,, \tag{131}$$

$$\phi_\epsilon \;=\; 43.52(5)^\circ\,, \tag{132}$$

$$\Delta M_K \;\equiv\; M_{K_L} - M_{K_S} \;=\; 3.484(6)\times 10^{-12}\,\mathrm{MeV}\,, \tag{133}$$

$$\Delta\Gamma_K \;\equiv\; \Gamma_{K_S} - \Gamma_{K_L} \;=\; 7.3382(33)\times 10^{-12}\,\mathrm{MeV}\,, \tag{134}$$

where the latter three measurements have been obtained by imposing CPT symmetry.

We will start by discussing the short-distance effects (i) since they provide the dominant contribution to ϵ_K. To lowest order in the electroweak theory, the contribution to the $K^0-\bar{K}^0$ oscillations arises from so-called box diagrams, in which two W bosons and two "up-type" quarks (i.e., up, charm, top) are exchanged between the constituent down and strange quarks of the K mesons. The loop integration of the box diagrams can be performed exactly. In the limit of vanishing external momenta and external quark masses, the result can be identified with an effective four-fermion interaction, expressed in terms of the effective Hamiltonian

$$\mathcal{H}_{\rm eff}^{\Delta S=2} \;=\; \frac{G_F^2 M_{\rm W}^2}{16\pi^2}\mathcal{F}^0 Q^{\Delta S=2} \;+\; {\rm h.c.}\;. \tag{135}$$

In this expression, G_F is the Fermi coupling, $M_{\rm W}$ the W-boson mass, and

$$Q^{\Delta S=2} = [\bar{s}\gamma_\mu(1-\gamma_5)d]\,[\bar{s}\gamma_\mu(1-\gamma_5)d] \equiv O_{\rm VV+AA} - O_{\rm VA+AV}\;, \tag{136}$$

is a dimension-six, four-fermion operator. The function $\mathcal{F}^0$ is given by

$$\mathcal{F}^0 \;=\; \lambda_c^2 S_0(x_c) \;+\; \lambda_t^2 S_0(x_t) \;+\; 2\lambda_c\lambda_t S_0(x_c,x_t)\;, \tag{137}$$

where $\lambda_a = V_{as}^* V_{ad}$, and $a = c\,,t$ denotes a flavour index. The quantities $S_0(x_c)$, $S_0(x_t)$ and $S_0(x_c,x_t)$ with $x_c = m_c^2/M_{\rm W}^2$, $x_t = m_t^2/M_{\rm W}^2$ are the Inami-Lim functions [11], which express the basic electroweak loop contributions without QCD corrections. The contribution of the up quark, which is taken to be massless in this approach, has been taken into account by imposing the unitarity constraint $\lambda_u + \lambda_c + \lambda_t = 0$.

When strong interactions are included, $\Delta S = 2$ transitions can no longer be discussed at the quark level. Instead, the effective Hamiltonian must be considered between mesonic initial and final states. Since the strong coupling is large at typical hadronic scales, the resulting weak matrix element cannot be calculated in perturbation theory. The operator product expansion (OPE) does, however, factorize long- and short- distance effects. For energy scales below the charm threshold, the $K^0 - \bar{K}^0$ transition amplitude of the effective Hamiltonian can be expressed as

$$\begin{aligned}\langle \bar{K}^0|\mathcal{H}_{\rm eff}^{\Delta S=2}|K^0\rangle \;&=\; \frac{G_F^2 M_{\rm W}^2}{16\pi^2}\Big[\lambda_c^2 S_0(x_c)\eta_1 \;+\; \lambda_t^2 S_0(x_t)\eta_2 \;+\; 2\lambda_c\lambda_t S_0(x_c,x_t)\eta_3\Big] \\ \times \left(\frac{\bar{g}(\mu)^2}{4\pi}\right)^{-\gamma_0/(2\beta_0)} & \exp\left\{\int_0^{\bar{g}(\mu)} dg\,\left(\frac{\gamma(g)}{\beta(g)} \;+\; \frac{\gamma_0}{\beta_0 g}\right)\right\}\langle \bar{K}^0|Q_{\rm R}^{\Delta S=2}(\mu)|K^0\rangle \;+\; {\rm h.c.}\;,\end{aligned} \tag{138}$$

where $\bar{g}(\mu)$ and $Q_{\rm R}^{\Delta S=2}(\mu)$ are the renormalized gauge coupling and four-fermion operator in some renormalization scheme. The factors η_1, η_2 and η_3 depend on the renormalized coupling $\bar{g}$, evaluated at the various flavour thresholds m_t, m_b, m_c and $M_{\rm W}$, as required by the OPE and RG-running procedure that separate high- and low-energy contributions. Explicit expressions can be found in Refs. [3] and references therein, except that η_1 and η_3 have been calculated to NNLO in Refs. [12] and [13], respectively. We follow the same conventions for the RG equations as in Ref. [3]. Thus the Callan-Symanzik function and the anomalous dimension $\gamma(\bar{g})$ of $Q^{\Delta S=2}$ are defined by

$$\frac{d\bar{g}}{d\ln\mu} = \beta(\bar{g})\,, \qquad \frac{dQ_{\rm R}^{\Delta S=2}}{d\ln\mu} = -\gamma(\bar{g})\,Q_{\rm R}^{\Delta S=2}\;, \tag{139}$$

with perturbative expansions

$$\begin{aligned} \beta(g) &= -\beta_0 \frac{g^3}{(4\pi)^2} - \beta_1 \frac{g^5}{(4\pi)^4} - \cdots, \\ \gamma(g) &= \gamma_0 \frac{g^2}{(4\pi)^2} + \gamma_1 \frac{g^4}{(4\pi)^4} + \cdots. \end{aligned} \tag{140}$$

We stress that β_0, β_1 and γ_0 are universal, i.e., scheme independent. As for $K^0 - \bar{K}^0$ mixing, this is usually considered in the naive dimensional regularization (NDR) scheme of $\overline{\mathrm{MS}}$, and below we specify the perturbative coefficient γ_1 in that scheme:

$$\begin{aligned} \beta_0 &= \left\{\frac{11}{3}N - \frac{2}{3}N_f\right\}, \qquad \beta_1 = \left\{\frac{34}{3}N^2 - N_f\left(\frac{13}{3}N - \frac{1}{N}\right)\right\}, \\ \gamma_0 &= \frac{6(N-1)}{N}, \qquad \gamma_1 = \frac{N-1}{2N}\left\{-21 + \frac{57}{N} - \frac{19}{3}N + \frac{4}{3}N_f\right\}. \end{aligned} \tag{141}$$

Note that for QCD the above expressions must be evaluated for $N = 3$ colours, while N_f denotes the number of active quark flavours. As already stated, Eq. (138) is valid at scales below the charm threshold, after all heavier flavours have been integrated out, i.e., $N_f = 3$.

In Eq. (138), the terms proportional to η_1, η_2 and η_3, multiplied by the contributions containing $\bar{g}(\mu)^2$, correspond to the Wilson coefficient of the OPE, computed in perturbation theory. Its dependence on the renormalization scheme and scale μ is canceled by that of the weak matrix element $\langle \bar{K}^0 | Q_{\mathrm{R}}^{\Delta S=2}(\mu) | K^0 \rangle$. The latter corresponds to the long-distance effects of the effective Hamiltonian and must be computed nonperturbatively. For historical, as well as technical reasons, it is convenient to express it in terms of the B-parameter B_K, defined as

$$B_K(\mu) = \frac{\left\langle \bar{K}^0 \left| Q_{\mathrm{R}}^{\Delta S=2}(\mu) \right| K^0 \right\rangle}{\frac{8}{3} f_K^2 m_K^2}. \tag{142}$$

The four-quark operator $Q^{\Delta S=2}(\mu)$ is renormalized at scale μ in some regularization scheme, for instance, NDR-$\overline{\mathrm{MS}}$. Assuming that $B_K(\mu)$ and the anomalous dimension $\gamma(g)$ are both known in that scheme, the renormalization group independent (RGI) B-parameter $\hat{B}_K$ is related to $B_K(\mu)$ by the exact formula

$$\hat{B}_K = \left(\frac{\bar{g}(\mu)^2}{4\pi}\right)^{-\gamma_0/(2\beta_0)} \exp\left\{ \int_0^{\bar{g}(\mu)} dg \left(\frac{\gamma(g)}{\beta(g)} + \frac{\gamma_0}{\beta_0 g}\right)\right\} B_K(\mu). \tag{143}$$

At NLO in perturbation theory the above reduces to

$$\hat{B}_K = \left(\frac{\bar{g}(\mu)^2}{4\pi}\right)^{-\gamma_0/(2\beta_0)} \left\{1 + \frac{\bar{g}(\mu)^2}{(4\pi)^2}\left[\frac{\beta_1\gamma_0 - \beta_0\gamma_1}{2\beta_0^2}\right]\right\} B_K(\mu). \tag{144}$$

To this order, this is the scale-independent product of all μ-dependent quantities in Eq. (138).

Lattice-QCD calculations provide results for $B_K(\mu)$. These results are, however, usually obtained in intermediate schemes other than the continuum $\overline{\mathrm{MS}}$ scheme used to calculate the Wilson coefficients appearing in Eq. (138). Examples of intermediate schemes are the RI/MOM scheme [14] (also dubbed the "Rome-Southampton method") and the Schrödinger functional (SF) scheme [15]. These schemes are used as they allow a nonperturbative renormalization of the four-fermion operator, using an auxiliary lattice simulation. This allows $B_K(\mu)$ to be calculated with percent-level accuracy, as described below.

In order to make contact with phenomenology, however, and in particular to use the results presented above, one must convert from the intermediate scheme to the $\overline{\mathrm{MS}}$ scheme or to the RGI quantity $\hat{B}_K$. This conversion relies on one or 2-loop perturbative matching calculations, the truncation errors in which are, for many recent calculations, the dominant source of error in $\hat{B}_K$ (see, for instance, Refs. [16–20]). While this scheme-conversion error is not, strictly speaking, an error of the lattice calculation itself, it must be included in results for the quantities of phenomenological interest, namely, $B_K(\overline{\mathrm{MS}}, 2\,\mathrm{GeV})$ and $\hat{B}_K$. Incidentally, we remark that this truncation error is estimated in different ways and that its relative contribution to the total error can considerably differ among the various lattice calculations. We note that this error can be minimized by matching between the intermediate scheme and $\overline{\mathrm{MS}}$ at as large a scale μ as possible (so that the coupling which determines the rate of convergence is minimized). Recent calculations have pushed the matching μ up to the range $3 - 3.5\,\mathrm{GeV}$. This is possible because of the use of nonperturbative RG running determined on the lattice [17, 19, 21]. The Schrödinger functional offers the possibility to run nonperturbatively to scales $\mu \sim M_{\mathrm{W}}$ where the truncation error can be safely neglected. However, so far this has been applied only for two flavours for B_K in Ref. [22] and for the case of the BSM bag parameters in Ref. [23], see more details in Sec. 6.3.

Perturbative truncation errors in Eq. (138) also affect the Wilson coefficients η_1, η_2 and η_3. It turns out that the largest uncertainty arises from the charm quark contribution $\eta_1 = 1.87(76)$ [12]. Although it is now calculated at NNLO, the series shows poor convergence. The net effect from the uncertainty on η_1 on the amplitude in Eq. (138) is larger than that of present lattice calculations of B_K.

We will now proceed to discuss the remaining contributions to ϵ_K in Eq. (129). An analytical estimate of the leading contribution from $\mathrm{Im}M_{12}^{\mathrm{LD}}$ based on χPT, shows that it is approximately proportional to $\xi \equiv \mathrm{Im}(A_0)/\mathrm{Re}(A_0)$ so that Eq. (129) can be written as follows [8, 9]

$$\epsilon_K \;=\; \exp(i\phi_\epsilon)\;\sin(\phi_\epsilon)\left[\frac{\mathrm{Im}(M_{12}^{\mathrm{SD}})}{\Delta M_K} \;+\; \rho\,\xi\,\right], \tag{145}$$

where the deviation of ρ from one parameterizes the long-distance effects in $\mathrm{Im}M_{12}$.

An estimate of ξ has been obtained from a direct evaluation of the ratio of amplitudes $\mathrm{Im}(A_0)/\mathrm{Re}(A_0)$ where $\mathrm{Im}(A_0)$ is determined from a lattice-QCD computation [24] at one value of the lattice spacing, while $\mathrm{Re}(A_0) \simeq |A_0|$ and the value $|A_0| = 3.320(2)\times 10^{-7}$ GeV are used based on the relevant experimental input [10] from the decay to two pions. This leads to a result for ξ with a rather large relative error,

$$\xi = -0.6(5)\cdot 10^{-4}. \tag{146}$$

A more precise estimate can be been obtained through a lattice-QCD computation of the ratio of amplitudes $\mathrm{Im}(A_2)/\mathrm{Re}(A_2)$ [25] where the continuum limit result is based on data at two values of the lattice spacing; A_2 denotes the $\Delta I = 3/2$ $K \to \pi\pi$ decay amplitude. For the computation of ξ, the experimental values of $\mathrm{Re}(\epsilon'/\epsilon)$, $|\epsilon_K|$ and $\omega = \mathrm{Re}(A_2)/\mathrm{Re}(A_0)$ have been used. The result for ξ reads

$$\xi = -1.6(2)\cdot 10^{-4}. \tag{147}$$

A phenomenological estimate can also be obtained from the relationship of ξ to $\mathrm{Re}(\epsilon'/\epsilon)$, using the experimental value of the latter and further assumptions concerning the estimate of hadronic contributions. The corresponding value of ξ reads [8, 9]

$$\xi = -6.0(1.5) \cdot 10^{-2}\sqrt{2}\,|\epsilon_K| = -1.9(5) \cdot 10^{-4}. \tag{148}$$

We note that the use of the experimental value for $\mathrm{Re}(\epsilon'/\epsilon)$ is based on the assumption that it is free from New Physics contributions. The value of ξ can then be combined with a χPT-based estimate for the long-range contribution, $\rho = 0.6(3)$ [9]. Overall, the combination $\rho\xi$ appearing in Eq. (145) leads to a suppression of the SM prediction of $|\epsilon_K|$ by about 3(2)% relative to the experimental measurement of $|\epsilon_K|$ given in Eq. (131), regardless of whether the phenomenological estimate of ξ [see Eq. (148)] or the most precise lattice result [see Eq. (147)] are used. The uncertainty in the suppression factor is dominated by the error on ρ. Although this is a small correction, we note that its contribution to the error of ϵ_K is larger than that arising from the value of B_K reported below.

Efforts are under way to compute the long-distance contributions to ϵ_K [26] and to the $K_L - K_S$ mass difference in lattice QCD [27–30]. However, the results are not yet precise enough to improve the accuracy in the determination of the parameter ρ.

The lattice-QCD study of $K \to \pi\pi$ decays provides crucial input to the SM prediction of ϵ_K. Besides the RBC-UKQCD collaboration programme [24, 25] using domain-wall fermions, an approach based on improved Wilson fermions [31, 32] has presented a determination of the $K \to \pi\pi$ decay amplitudes, A_0 and A_2, at unphysical quark masses. A first proposal aiming at the inclusion of electromagnetism in lattice-QCD calculations of these decays was reported in Ref. [33]. For an ongoing analysis of the scaling with the number of colours of $K \to \pi\pi$ decay amplitudes using lattice-QCD computations, we refer to Refs. [34, 35].

Finally, we notice that ϵ_K receives a contribution from $|V_{cb}|$ through the λ_t parameter in Eq. (137). The present uncertainty on $|V_{cb}|$ has a significant impact on the error of ϵ_K [see, e.g., Ref. [36] and a recent update [37]].

6.2 Lattice computation of B_K

Lattice calculations of B_K are affected by the same systematic effects discussed in previous sections. However, the issue of renormalization merits special attention. The reason is that the multiplicative renormalizability of the relevant operator $Q^{\Delta S=2}$ is lost once the regularized QCD action ceases to be invariant under chiral transformations. For Wilson fermions, $Q^{\Delta S=2}$ mixes with four additional dimension-six operators, which belong to different representations of the chiral group, with mixing coefficients that are finite functions of the gauge coupling. This complicated renormalization pattern was identified as the main source of systematic error in earlier, mostly quenched calculations of B_K with Wilson quarks. It can be bypassed via the implementation of specifically designed methods, which are either based on Ward identities [38] or on a modification of the Wilson quark action, known as twisted mass QCD [39–41].

An advantage of staggered fermions is the presence of a remnant $U(1)$ chiral symmetry. However, at nonvanishing lattice spacing, the symmetry among the extra unphysical degrees of freedom (tastes) is broken. As a result, mixing with other dimension-six operators cannot be avoided in the staggered formulation, which complicates the determination of the B-parameter. In general, taste conserving mixings are implemented directly in the lattice computation of the matrix element. The effects of the broken taste symmetry are usually treated through an effective field theory, staggered Chiral Perturbation Theory (SχPT) [42, 43], parameterizing the quark-mass and lattice-spacing dependences.

Fermionic lattice actions based on the Ginsparg-Wilson relation [44] are invariant under the chiral group, and hence four-quark operators such as $Q^{\Delta S=2}$ renormalize multiplicatively. However, depending on the particular formulation of Ginsparg-Wilson fermions, residual chiral symmetry breaking effects may be present in actual calculations. For instance, in the case of domain-wall fermions, the finiteness of the extra 5th dimension implies that the decoupling of modes with different chirality is not exact, which produces a residual nonzero quark mass in the chiral limit. Furthermore, whether a significant mixing with dimension-six operators of different chirality is induced must be investigated on a case-by-case basis.

The only existing lattice-QCD calculation of B_K with $N_f = 2+1+1$ dynamical quarks [45] was reviewed in the FLAG 16 report. Considering that no direct evaluation of the size of the excess of charm quark effects included in $N_f = 2 + 1 + 1$ computations of B_K has appeared since then, we wish to reiterate a discussion about a few related conceptual issues.

As described in Sec. 6.1, kaon mixing is expressed in terms of an effective four-quark interaction $Q^{\Delta S=2}$, considered below the charm threshold. When the matrix element of $Q^{\Delta S=2}$ is evaluated in a theory that contains a dynamical charm quark, the resulting estimate for B_K must then be matched to the three-flavour theory that underlies the effective four-quark interaction.[1] In general, the matching of $2+1$-flavour QCD with the theory containing $2 + 1 + 1$ flavours of sea quarks below the charm threshold can be accomplished by adjusting the coupling and quark masses of the $N_f = 2 + 1$ theory so that the two theories match at energies $E < m_c$. The corrections associated with this matching are of order $(E/m_c)^2$, since the subleading operators have dimension eight [46].

When the kaon mixing amplitude is considered, the matching also involves the relation between the relevant box graphs and the effective four-quark operator. In this case, corrections of order $(E/m_c)^2$ arise not only from the charm quarks in the sea, but also from the valence sector, since the charm quark propagates in the box diagrams. We note that the original derivation of the effective four-quark interaction is valid up to corrections of order $(E/m_c)^2$. The kaon mixing amplitudes evaluated in the $N_f = 2 + 1$ and $2 + 1 + 1$ theories are thus subject to corrections of the same order in E/m_c as the derivation of the conventional four-quark interaction.

Regarding perturbative QCD corrections at the scale of the charm quark mass on the amplitude in Eq. (138), the uncertainty on η_1 and η_3 factors is of $\mathcal{O}(\alpha_s(m_c)^3)$ [12, 13], while that on η_2 is of $\mathcal{O}(\alpha_s(m_c)^2)$ [47]. On the other hand, a naive power counting argument suggests that the corrections of order $(E/m_c)^2$ due to dynamical charm-quark effects in the matching of the amplitudes are suppressed by powers of $\alpha_s(m_c)$ and by a factor of $1/N_c$. It is therefore essential that any forthcoming calculation of B_K with $N_f = 2 + 1 + 1$ flavours addresses properly the size of these residual dynamical charm effects in a quantitative way.

Another issue in this context is how the lattice scale and the physical values of the quark masses are determined in the $2 + 1$ and $2 + 1 + 1$ flavour theories. Here it is important to consider in which way the quantities used to fix the bare parameters are affected by a dynamical charm quark. Apart from a brief discussion in Ref. [45], these issues have not yet been directly addressed in the literature.[2] Given the hierarchy of scales between the charm quark mass and that of B_K, we expect these errors to be modest, but a more quantitative understanding is needed as statistical errors on B_K are reduced. Within this review we will

[1] We thank Martin Lüscher for an interesting discussion on this issue.

[2] The nonperturbative studies with two heavy mass-degenerate quarks in Refs. [48, 49] indicate that dynamical charm-quark effects in low-energy hadronic observables are considerably smaller than the expectation from a naive power counting in terms of $\alpha_s(m_c)$.

not discuss this issue further. However, we wish to point out that the present discussion also applies to $N_f = 2+1+1$ computations of the kaon BSM B-parameters discussed in Sec. 6.3.

A compilation of results with $N_f = 2, 2+1$ and $2+1+1$ flavours of dynamical quarks is shown in Tabs. 27 and 28, as well as Fig. 18. An overview of the quality of systematic error studies is represented by the colour coded entries in Tabs. 27 and 28. In Appendix B.4 we gather the simulation details and results that have appeared since the previous FLAG review [50]. The values of the most relevant lattice parameters, and comparative tables on the various estimates of systematic errors are also collected.

Some of the groups whose results are listed in Tabs. 27 and 28 do not quote results for both $B_K(\overline{\rm MS}, 2\,{\rm GeV})$—which we denote by the shorthand B_K from now on—and $\hat{B}_K$. This concerns Refs. [51, 52] for $N_f = 2$, Refs.[16, 17, 19, 20] for $2+1$ and Ref. [45] for $2+1+1$ flavours. In these cases we perform the conversion ourselves by evaluating the proportionality factor in Eq. (144) using perturbation theory at NLO at a renormalization scale $\mu = 2\,{\rm GeV}$. For $N_f = 2+1$, by using the world average value $\Lambda^{(3)}_{\overline{\rm MS}} = 332\,{\rm MeV}$ from PDG [10] and the 4-loop β-function we obtain, $\hat{B}_K/B_K = 1.369$ in the three-flavour theory. Had we used the 5-loop β-function we would get $\hat{B}_K/B_K = 1.373$. If we use instead the average lattice results from Sec. 9 of the present FLAG report, $\Lambda^{(3)}_{\overline{\rm MS}} = 343\,{\rm MeV}$, together with the four and 5-loop β-function, we obtain $\hat{B}_K/B_K = 1.365$ and $\hat{B}_K/B_K = 1.369$, respectively. In FLAG 16, we used $\hat{B}_K/B_K = 1.369$ based on the 2014 edition of the PDG [53]. The relative deviations among these various estimates is below the 3 permille level and amounts to a tiny fraction of the uncertainty on the average value of the B-parameter. We have therefore used in this edition the value, $\hat{B}_K/B_K = 1.369$, which was also used in FLAG 16. The same value for the conversion factor has also been applied to the result computed in QCD with $N_f = 2+1+1$ flavours of dynamical quarks [45].

In two-flavour QCD one can insert into the NLO expressions for α_s the estimate $\Lambda^{(2)}_{\overline{\rm MS}} = 330\,{\rm MeV}$, which is the average value for $N_f = 2$ obtained in Sec. 9, and get $\hat{B}_K/B_K = 1.365$ and $\hat{B}_K/B_K = 1.368$ for running with four and 5-loop β-function, respectively. We again note that the difference between the conversion factors for $N_f = 2$ and $N_f = 2+1$ will produce a negligible ambiguity, which, in any case, is well below the overall uncertainties in Refs. [51, 52]. We have therefore chosen to apply the conversion factor of 1.369 not only to results obtained for $N_f = 2+1$ flavours but also to the two-flavour theory (in cases where only one of $\hat{B}_K$ and B_K are quoted). We have indicated explicitly in Tab. 28 in which way the conversion factor 1.369 has been applied to the results of Refs. [51, 52]. We wish to encourage authors to provide both $\hat{B}_K$ and B_K together with the values of the parameters appearing in the perturbative running.

We discuss here one recent result for the kaon B-parameter reported by the RBC/UKQCD collaboration, RBC/UKQCD 16 [54], where $N_f = 2+1$ dynamical quarks have been employed. For a detailed description of previous calculations—and in particular those considered in the computation of the average values—we refer the reader to the FLAG 16 [50] and FLAG 13 [55] reports.

Collaboration	Ref.	N_f	publication status	continuum extrapolation	chiral extrapolation	finite volume	renormalization	running	$B_K(\overline{\mathrm{MS}}, 2\,\mathrm{GeV})$	$\hat{B}_K$
ETM 15	[45]	2+1+1	A	★	○	○	★	a	0.524(13)(12)	$0.717(18)(16)^1$
RBC/UKQCD 16	[54]	2+1	A	○	○	○	★	b	$0.543(9)(13)^2$	$0.744(13)(18)^3$
SWME 15A	[20]	2+1	A	★	○	★	○‡	–	0.537(4)(26)	$0.735(5)(36)^4$
RBC/UKQCD 14B	[19]	2+1	A	★	★	○	★	b	$0.5478(18)(110)^2$	0.7499(24)(150)
SWME 14	[18]	2+1	A	★	○	★	○‡	–	0.5388(34)(266)	0.7379(47)(365)
SWME 13A	[56]	2+1	A	★	○	★	○‡	–	0.537(7)(24)	0.735(10)(33)
SWME 13	[57]	2+1	C	★	○	★	○‡	–	0.539(3)(25)	0.738(5)(34)
RBC/UKQCD 12A	[17]	2+1	A	○	★	○	★	b	$0.554(8)(14)^2$	0.758(11)(19)
Laiho 11	[16]	2+1	C	★	○	○	★	–	0.5572(28)(150)	$0.7628(38)(205)^4$
SWME 11A	[58]	2+1	A	★	○	○	○‡	–	0.531(3)(27)	0.727(4)(38)
BMW 11	[21]	2+1	A	★	★	★	★	c	0.5644(59)(58)	0.7727(81)(84)
RBC/UKQCD 10B	[59]	2+1	A	○	○	★	★	d	0.549(5)(26)	0.749(7)(26)
SWME 10	[60]	2+1	A	★	○	○	○	–	0.529(9)(32)	0.724(12)(43)
Aubin 09	[61]	2+1	A	○	○	○	★	–	0.527(6)(21)	0.724(8)(29)

‡ The renormalization is performed using perturbation theory at one loop, with a conservative estimate of the uncertainty.

a B_K is renormalized nonperturbatively at scales $1/a \sim 2.2 - 3.3\,\mathrm{GeV}$ in the $N_f = 4$ RI/MOM scheme using two different lattice momentum scale intervals, the first around $1/a$ while the second around 3.5 GeV. The impact of the two ways to the final result is taken into account in the error budget. Conversion to $\overline{\mathrm{MS}}$ is at 1-loop at 3 GeV.

b B_K is renormalized nonperturbatively at a scale of 1.4 GeV in two RI/SMOM schemes for $N_f = 3$, and then run to 3 GeV using a nonperturbatively determined step-scaling function. Conversion to $\overline{\mathrm{MS}}$ is at 1-loop order at 3 GeV.

c B_K is renormalized and run nonperturbatively to a scale of 3.5 GeV in the RI/MOM scheme. At the same scale conversion at one loop to $\overline{\mathrm{MS}}$ is applied. Nonperturbative and NLO perturbative running agrees down to scales of 1.8 GeV within statistical uncertainties of about 2%.

d B_K is renormalized nonperturbatively at a scale of 2 GeV in two RI/SMOM schemes for $N_f = 3$, and then run to 3 GeV using a nonperturbatively determined step-scaling function. Conversion to $\overline{\mathrm{MS}}$ is at 1-loop order at 3 GeV.

1 $B_K(\overline{\mathrm{MS}}, 2\,\mathrm{GeV})$ and $\hat{B}_K$ are related using the conversion factor 1.369, i.e., the one obtained with $N_f = 2+1$.

2 $B_K(\overline{\mathrm{MS}}, 2\,\mathrm{GeV})$ is obtained from the estimate for $\hat{B}_K$ using the conversion factor 1.369.

3 $\hat{B}_K$ is obtained from $B_K(\overline{\mathrm{MS}}, 3\,\mathrm{GeV})$ using the conversion factor employed in Ref. [19].

4 $\hat{B}_K$ is obtained from the estimate for $B_K(\overline{\mathrm{MS}}, 2\,\mathrm{GeV})$ using the conversion factor 1.369.

Table 27: Results for the kaon B-parameter in QCD with $N_f = 2+1+1$ and $N_f = 2+1$ dynamical flavours, together with a summary of systematic errors. Any available information about nonperturbative running is indicated in the column "running", with details given at the bottom of the table.

Collaboration	Ref.	N_f	publication status	continuum extrapolation	chiral extrapolation	finite volume	renormalization	running	$B_K(\overline{\mathrm{MS}}, 2\,\mathrm{GeV})$	$\hat{B}_K$
ETM 12D	[52]	2	A	★	○	○	★	e	0.531(16)(9)	0.727(22)(12)[1]
ETM 10A	[51]	2	A	★	○	○	★	f	0.533(18)(12)[1]	0.729(25)(17)

e B_K is renormalized nonperturbatively at scales $1/a \sim 2-3.7\,\mathrm{GeV}$ in the $N_f = 2$ RI/MOM scheme. In this scheme, nonperturbative and NLO perturbative running are shown to agree from 4 GeV down to 2 GeV to better than 3% [51, 62].

f B_K is renormalized nonperturbatively at scales $1/a \sim 2-3\,\mathrm{GeV}$ in the $N_f = 2$ RI/MOM scheme. In this scheme, nonperturbative and NLO perturbative running are shown to agree from 4 GeV down to 2 GeV to better than 3% [51, 62].

[1] $B_K(\overline{\mathrm{MS}}, 2\,\mathrm{GeV})$ and $\hat{B}_K$ are related using the conversion factor 1.369, i.e., the one obtained with $N_f = 2+1$.

Table 28: Results for the kaon B-parameter in QCD with $N_f = 2$ dynamical flavours, together with a summary of systematic errors. Any available information about nonperturbative running is indicated in the column "running", with details given at the bottom of the table.

In Ref. [54], RBC/UKQCD presented a determination of B_K obtained as part of their study of kaon mixing in extensions of the SM. In this calculation two values of the lattice spacing, $a \simeq 0.11$ and 0.08 fm, are used, employing ensembles generated using the Iwasaki gauge action and the Shamir domain-wall fermionic action. The lattice volumes are $24^3 \times 64 \times 16$ for the coarse and $32^3 \times 64 \times 16$ for the fine lattice spacing. The lowest simulated values for the pseudoscalar mass are about 340 MeV and 300 MeV, respectively. The renormalization of four-quark operators was performed nonperturbatively in two RI-SMOM schemes, namely, $(\not{q}, \not{q})$ and (γ_μ, γ_μ), where the latter was used for the final estimate of B_K. While the procedure to determine B_K is very similar to RBC/UKQCD 14B, the calculation in RBC/UKQCD 16 [54] is based only on a subset of the ensembles studied in Ref. [19]. Therefore, the result for B_K reported in Ref. [54] can neither be considered an update of RBC/UKQCD 14B, nor an independent new result.

We now describe our procedure for obtaining global averages. The rules of Sec. 2.1 stipulate that results free of red tags and published in a refereed journal may enter an average. Papers that at the time of writing are still unpublished but are obvious updates of earlier published results can also be taken into account.

There is only one result for $N_f = 2+1+1$, computed by the ETM collaboration [45]. Since it is free of red tags, it qualifies as the currently best global estimate, i.e.,

$$N_f = 2+1+1: \quad \hat{B}_K = 0.717(18)(16)\,, \quad B_K^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.524(13)(12) \quad \text{Ref. [45]}. \tag{149}$$

The bulk of results for the kaon B-parameter has been obtained for $N_f = 2+1$. As in the previous editions of the FLAG review [50, 55] we include the results from SWME [18, 20, 56], despite the fact that nonperturbative information on the renormalization factors is not available. Instead, the matching factor has been determined in perturbation theory at one loop, but with a sufficiently conservative error of 4.4%. As described above, the result in RBC/UKQCD 16 [54] cannot be considered an update of the earlier estimate in RBC/UKQCD 14B, and hence it is not included in the FLAG average.

Thus, for $N_f = 2{+}1$ our global average is based on the results of BMW 11 [21], Laiho 11 [16], RBC/UKQCD 14B [19] and SWME 15A [20]. The last three are the latest updates from a series of calculations by the same collaborations. Our procedure is as follows: in a first step statistical and systematic errors of each individual result for the RGI B-parameter, $\hat{B}_K$, are combined in quadrature. Next, a weighted average is computed from the set of results. For the final error estimate we take correlations between different collaborations into account. To this end we note that we consider the statistical and finite-volume errors of SWME 15A and Laiho 11 to be correlated, since both groups use the Asqtad ensembles generated by the MILC collaboration. Laiho 11 and RBC/UKQCD 14B both use domain-wall quarks in the valence sector and also employ similar procedures for the nonperturbative determination of matching factors. Hence, we treat the quoted renormalization and matching uncertainties by the two groups as correlated. After constructing the global covariance matrix according to Schmelling [63], we arrive at

$$N_f = 2+1: \qquad \hat{B}_K = 0.7625(97) \qquad \text{Refs. [16, 19–21]}, \tag{150}$$

with $\chi^2/\text{dof} = 0.675$. After applying the NLO conversion factor $\hat{B}_K/B_K^{\overline{\text{MS}}}(2\,\text{GeV}) = 1.369$, this translates into

$$N_f = 2+1: \qquad B_K^{\overline{\text{MS}}}(2\,\text{GeV}) = 0.5570(71) \qquad \text{Refs. [16, 19–21]}. \tag{151}$$

Note that the statistical errors of each calculation entering the global average are small enough to make their results statistically incompatible. It is only because of the relatively large systematic errors that the weighted average produces a value of $\mathcal{O}(1)$ for the reduced χ^2.

Passing over to describing the results computed for $N_f = 2$ flavours, we note that there is only the set of results published in ETM 12D [52] and ETM 10A [51] that allow for an extensive investigation of systematic uncertainties. We identify the result from ETM 12D [52], which is an update of ETM 10A, with the currently best global estimate for two-flavour QCD, i.e.,

$$N_f = 2: \quad \hat{B}_K = 0.727(22)(12), \quad B_K^{\overline{\text{MS}}}(2\,\text{GeV}) = 0.531(16)(19) \quad \text{Ref. [52]}. \tag{152}$$

The result in the $\overline{\text{MS}}$ scheme has been obtained by applying the same conversion factor of 1.369 as in the three-flavour theory.

6.3 Kaon BSM B-parameters

We now report on lattice results concerning the matrix elements of operators that encode the effects of physics beyond the Standard Model (BSM) to the mixing of neutral kaons. In this theoretical framework both the SM and BSM contributions add up to reproduce the experimentally observed value of ϵ_K. Since BSM contributions involve heavy but unobserved particles they are short-distance dominated. The effective Hamiltonian for generic $\Delta S = 2$ processes including BSM contributions reads

$$\mathcal{H}_{\text{eff,BSM}}^{\Delta S=2} = \sum_{i=1}^{5} C_i(\mu) Q_i(\mu), \tag{153}$$

where Q_1 is the four-quark operator of Eq. (136) that gives rise to the SM contribution to ϵ_K. In the so-called SUSY basis introduced by Gabbiani et al. [64] the operators $Q_2, \ldots, Q_5$

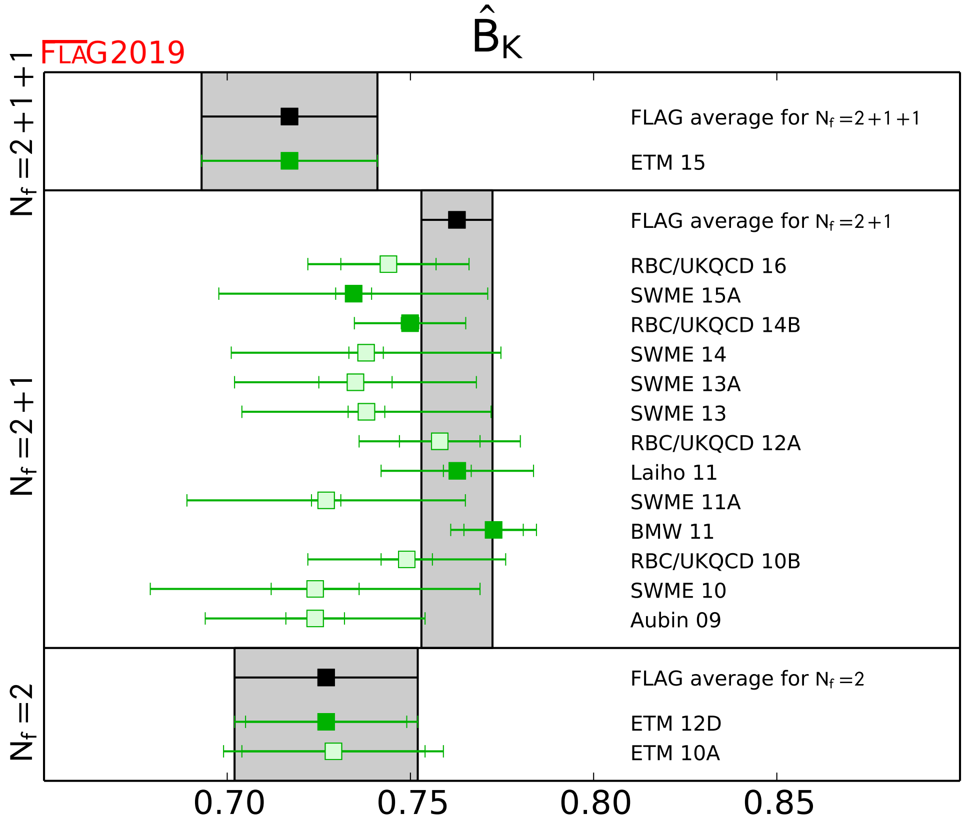

Figure 18: Recent unquenched lattice results for the RGI B-parameter $\hat{B}_K$. The grey bands indicate our global averages described in the text. For $N_f = 2+1+1$ and $N_f = 2$ the global estimate coincide with the results by ETM 12D and ETM 10A, respectively.

read [3]

$$\begin{aligned}
Q_2 &= \left(\bar{s}^a(1-\gamma_5)d^a\right)\left(\bar{s}^b(1-\gamma_5)d^b\right),\\
Q_3 &= \left(\bar{s}^a(1-\gamma_5)d^b\right)\left(\bar{s}^b(1-\gamma_5)d^a\right),\\
Q_4 &= \left(\bar{s}^a(1-\gamma_5)d^a\right)\left(\bar{s}^b(1+\gamma_5)d^b\right),\\
Q_5 &= \left(\bar{s}^a(1-\gamma_5)d^b\right)\left(\bar{s}^b(1+\gamma_5)d^a\right),
\end{aligned} \tag{154}$$

where a and b denote colour indices. In analogy to the case of B_K one then defines the B-parameters of $Q_2, \ldots, Q_5$ according to

$$B_i(\mu) = \frac{\left\langle \bar{K}^0 \left| Q_i(\mu) \right| K^0 \right\rangle}{N_i \left\langle \bar{K}^0 \left| \bar{s}\gamma_5 d \right| 0 \right\rangle \left\langle 0 \left| \bar{s}\gamma_5 d \right| K^0 \right\rangle}, \quad i = 2, \ldots, 5. \tag{155}$$

The factors $\{N_2, \ldots, N_5\}$ are given by $\{-5/3, 1/3, 2, 2/3\}$, and it is understood that $B_i(\mu)$ is specified in some renormalization scheme, such as $\overline{\text{MS}}$ or a variant of the regularization-independent momentum subtraction (RI-MOM) scheme.

[3] Thanks to QCD parity invariance lattice computations for three more dimension-six operators, whose parity conserving parts coincide with the corresponding parity conserving contributions of the operators Q_1, Q_2 and Q_3, can be ignored.

The SUSY basis has been adopted in Refs. [45, 52, 54, 65]. Alternatively, one can employ the chiral basis of Buras, Misiak and Urban [66]. The SWME collaboration prefers the latter since the anomalous dimension that enters the RG running has been calculated to two loops in perturbation theory [66]. Results obtained in the chiral basis can be easily converted to the SUSY basis via

$$B_3^{\rm SUSY} = \tfrac{1}{2}\left(5B_2^{\rm chiral} - 3B_3^{\rm chiral}\right). \tag{156}$$

The remaining B-parameters are the same in both bases. In the following we adopt the SUSY basis and drop the superscript.

Older quenched results for the BSM B-parameters can be found in Refs. [67–69]. For a nonlattice approach to get estimates for the BSM B-parameters see Ref. [70].

Estimates for $B_2, \dots, B_5$ have been reported for QCD with $N_f = 2$ (ETM 12D [52]), $N_f = 2+1$ (RBC/UKQCD 12E [65], SWME 13A [56], SWME 14C [71], SWME 15A [20], RBC/UKQCD 16 [54, 72]) and $N_f = 2+1+1$ (ETM 15 [45]) flavours of dynamical quarks. They are listed and compared in Tab. 29 and Fig. 19. In general one finds that the BSM B-parameters computed by different collaborations do not show the same level of consistency as the SM kaon mixing parameter B_K discussed previously. Control over systematic uncertainties (chiral and continuum extrapolations, finite-volume effects) in $B_2, \dots, B_5$ is expected to be at the same level as for B_K, as far as the results by ETM 12D, ETM 15 and SWME 15A are concerned. The calculation by RBC/UKQCD 12E has been performed at a single value of the lattice spacing and a minimum pion mass of 290 MeV. Thus, the results do not benefit from the same improvements regarding control over the chiral and continuum extrapolations as in the case of B_K [19].

The RBC/UKQCD collaboration has recently extended its calculation of BSM B-parameters [54, 72] for $N_f = 2+1$, by considering two values of the lattice spacing, $a \simeq 0.11$ and 0.08 fm, employing ensembles generated using the Iwasaki gauge action and the Shamir domain-wall fermionic action. The lattice volumes in the RBC/UKQCD 16 calculation are $24^3 \times 64 \times 16$ for the coarse and $32^3 \times 64 \times 16$ for the fine lattice spacing, while the lowest simulated values for the pseudoscalar mass are about 340 MeV and 300 MeV, respectively. As in the related calculation of B_K (RBC/UKQCD 14B [19]) the renormalization of four-quark operators was performed nonperturbatively in two RI-SMOM schemes, namely, $(\not{q}, \not{q})$ and (γ_μ, γ_μ), where the latter was used for the final estimates of $B_2, \dots, B_5$ quoted in Ref. [54]. By comparing the results obtained in the conventional RI-MOM and the two RI-SMOM schemes, RBC/UKQCD 16 report significant discrepancies for B_4 and B_5 in the $\overline{\rm MS}$ scheme at the scale of 3 GeV, which amount up to 2.8σ in the case of B_5. By contrast, the agreement for B_2 and B_3 determined for different intermediate scheme is much better. Based on these findings they claim that these discrepancies are due to uncontrolled systematics coming from the Goldstone boson pole subtraction procedure that is needed in the RI-MOM scheme, while pole subtraction effects are much suppressed in RI-SMOM thanks to the fact that the latter is based on nonexceptional momenta. The RBC/UKQCD collaboration has presented an ongoing study [73] in which simulations with two values of the lattice spacing at the physical point and with a third finer lattice spacing at $M_\pi = 234$ MeV are employed in order to obtain the BSM matrix elements in the continuum limit. Results are still preliminary.

The findings by RBC/UKQCD 16 [54, 72] provide evidence that the nonperturbative determination of the matching factors depends strongly on the details of the implementation of the Rome-Southampton method. The use of nonexceptional momentum configurations in the calculation of the vertex functions produces a significant modification of the renormalization factors, which affects the matching between $\overline{\mathrm{MS}}$ and the intermediate momentum subtraction scheme. This effect is most pronounced in B_4 and B_5. Furthermore, it can be noticed that the estimates for B_4 and B_5 from RBC/UKQCD 16 are much closer to those of SWME 15A. At the same time, the results for B_2 and B_3 obtained by ETM 15, SWME 15A and RBC/UKQCD 16 are in good agreement within errors.

Collaboration	Ref.	N_f	publication status	continuum extrapolation	chiral extrapolation	finite volume	renormalization	running	B_2	B_3	B_4	B_5
ETM 15	[45]	2+1+1	A	★	○	○	★	a	0.46(1)(3)	0.79(2)(5)	0.78(2)(4)	0.49(3)(3)
RBC/UKQCD 16	[54]	2+1	A	○	○	○	★	b	0.488(7)(17)	0.743(14)(65)	0.920(12)(16)	0.707(8)(44)
SWME 15A	[20]	2+1	A	★	○	★	○†	−	0.525(1)(23)	0.773(6)(35)	0.981(3)(62)	0.751(7)(68)
SWME 14C	[71]	2+1	C	★	○	★	○†	−	0.525(1)(23)	0.774(6)(64)	0.981(3)(61)	0.748(9)(79)
SWME 13A‡	[56]	2+1	A	★	○	★	○†	−	0.549(3)(28)	0.790(30)	1.033(6)(46)	0.855(6)(43)
RBC/ UKQCD 12E	[65]	2+1	A	■	○	★	★	b	0.43(1)(5)	0.75(2)(9)	0.69(1)(7)	0.47(1)(6)
ETM 12D	[52]	2	A	★	○	○	★	c	0.47(2)(1)	0.78(4)(2)	0.76(2)(2)	0.58(2)(2)

† The renormalization is performed using perturbation theory at one loop, with a conservative estimate of the uncertainty.

a B_i are renormalized nonperturbatively at scales $1/a \sim 2.2 - 3.3\,\mathrm{GeV}$ in the $N_f = 4$ RI/MOM scheme using two different lattice momentum scale intervals, with values around $1/a$ for the first and around 3.5 GeV for the second one. The impact of these two ways to the final result is taken into account in the error budget. Conversion to $\overline{\mathrm{MS}}$ is at one loop at 3 GeV.

b The B-parameters are renormalized nonperturbatively at a scale of 3 GeV.

c B_i are renormalized nonperturbatively at scales $1/a \sim 2 - 3.7\,\mathrm{GeV}$ in the $N_f = 2$ RI/MOM scheme using two different lattice momentum scale intervals, with values around $1/a$ for the first and around 3 GeV for the second one.

‡ The computation of B_4 and B_5 has been revised in Refs. [20] and [71].

Table 29: Results for the BSM B-parameters $B_2, \ldots, B_5$ in the $\overline{\mathrm{MS}}$ scheme at a reference scale of 3 GeV. Any available information on nonperturbative running is indicated in the column "running", with details given at the bottom of the table.

A nonperturbative computation of the running of the four-fermion operators contributing to the $B_2, \ldots, B_5$ parameters has been carried out with two dynamical flavours using the Schrödinger functional renormalization scheme [23]. Renormalization matrices of the operator basis are used to build step-scaling functions governing the continuum-limit running between hadronic and electroweak scales. A comparison to perturbative results using NLO (2-loops) for the four-fermion operator anomalous dimensions indicates that, at scales of about 3 GeV, nonperturbative effects can induce a sizeable contribution to the running.

A detailed look at the most recent calculations reported in ETM 15 [45], SWME 15A [20] and RBC/UKQCD 16 [54] reveals that cutoff effects appear to be larger for the BSM B-parameters compared to B_K. Depending on the details of the renormalization procedure and/or the fit ansatz for the combined chiral and continuum extrapolation, the results obtained at the coarsest lattice spacing differ by 15–30%. At the same time the available range of lattice spacings is typically much reduced compared to the corresponding calculations of B_K, as can be seen by comparing the quality criteria in Tabs. 27 and 29. Hence, the impact of the renormalization procedure and the continuum limit on the BSM B-parameters certainly requires further investigation.

Finally we present our estimates for the BSM B-parameters, quoted in the $\overline{\mathrm{MS}}$-scheme at scale 3 GeV. For $N_f = 2+1$ our estimate is given by the average between the results from SWME 15A and RBC/UKQCD 16, i.e.,

$$\begin{aligned}&N_f = 2+1: \\ &B_2 = 0.502(14), \quad B_3 = 0.766(32), \quad B_4 = 0.926(19), \quad B_5 = 0.720(38), \quad \text{Refs. [20, 54].}\end{aligned} \tag{157}$$

For $N_f = 2+1+1$ and $N_f = 2$, our estimates coincide with the ones by ETM 15 and ETM 12D, respectively, since there is only one computation for each case. Thus we quote

$$\begin{aligned}&N_f = 2+1+1: \\ &B_2 = 0.46(1)(3), \quad B_3 = 0.79(2)(4), \quad B_4 = 0.78(2)(4), \quad B_5 = 0.49(3)(3), \quad \text{Ref. [45],}\end{aligned} \tag{158}$$

$$\begin{aligned}&N_f = 2: \\ &B_2 = 0.47(2)(1), \quad B_3 = 0.78(4)(2), \quad B_4 = 0.76(2)(2), \quad B_5 = 0.58(2)(2), \quad \text{Ref. [52].}\end{aligned} \tag{159}$$

Based on the above discussion on the effects of employing different intermediate momentum subtraction schemes in the nonperturbative renormalization of the operators, the discrepancy for B_4 and B_5 results between $N_f = 2, 2+1+1$ and $N_f = 2+1$ computations should not be considered an effect associated with the number of dynamical flavours. As a closing remark, we encourage authors to provide the correlation matrix of the B_i parameters.

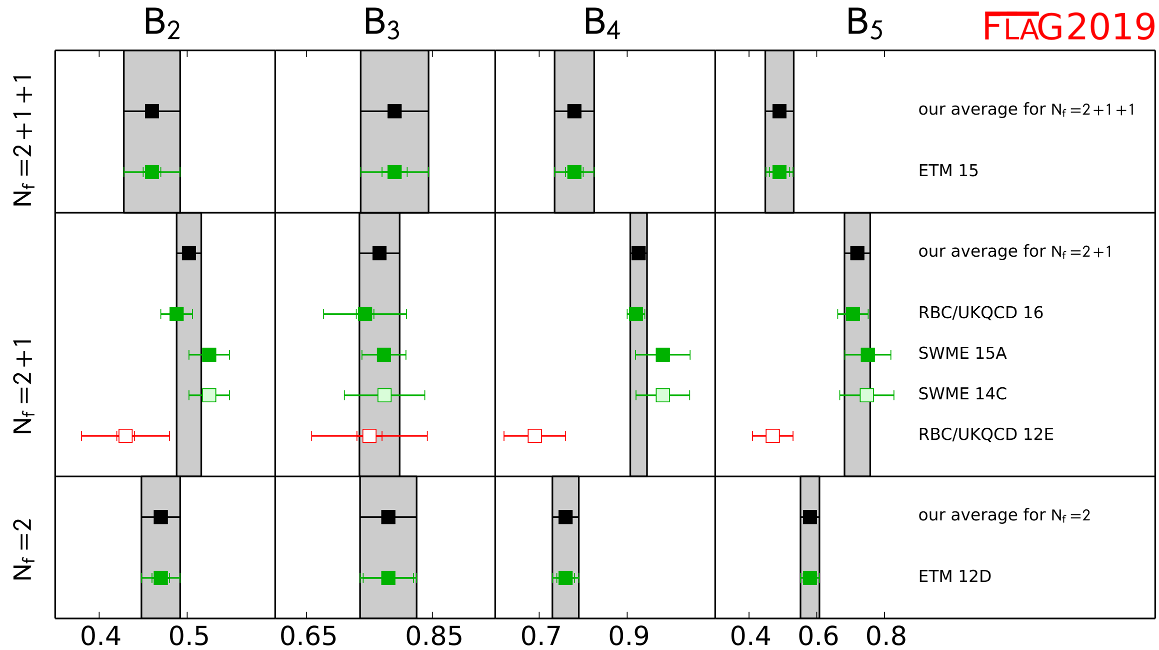

Figure 19: Lattice results for the BSM B-parameters defined in the $\overline{\mathrm{MS}}$ scheme at a reference scale of 3 GeV, see Tab. 29.

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