1 Introduction

Flavour physics provides an important opportunity for exploring the limits of the Standard Model of particle physics and for constraining possible extensions that go beyond it. As the LHC explores a new energy frontier and as experiments continue to extend the precision frontier, the importance of flavour physics will grow, both in terms of searches for signatures of new physics through precision measurements and in terms of attempts to construct the theoretical framework behind direct discoveries of new particles. Crucial to such searches for new physics is the ability to quantify strong-interaction effects. Large-scale numerical simulations of lattice QCD allow for the computation of these effects from first principles. The scope of the Flavour Lattice Averaging Group (FLAG) is to review the current status of lattice results for a variety of physical quantities that are important for flavour physics. Set up in November 2007, it comprises experts in Lattice Field Theory, Chiral Perturbation Theory and Standard Model phenomenology. Our aim is to provide an answer to the frequently posed question "What is currently the best lattice value for a particular quantity?" in a way that is readily accessible to those who are not expert in lattice methods. This is generally not an easy question to answer; different collaborations use different lattice actions (discretizations of QCD) with a variety of lattice spacings and volumes, and with a range of masses for the u- and d-quarks. Not only are the systematic errors different, but also the methodology used to estimate these uncertainties varies between collaborations. In the present work, we summarize the main features of each of the calculations and provide a framework for judging and combining the different results. Sometimes it is a single result that provides the "best" value; more often it is a combination of results from different collaborations. Indeed, the consistency of values obtained using different formulations adds significantly to our confidence in the results.

The first four editions of the FLAG review were made public in 2010 [1], 2013 [2], 2016 [3], and 2019 [4] (and will be referred to as FLAG 10, FLAG 13, FLAG 16, and FLAG 19, respectively). The fourth edition reviewed results related to both light (u-, d- and s-), and heavy (c- and b-) flavours. The quantities related to pion and kaon physics were light-quark masses, the form factor $f_+(0)$ arising in semileptonic $K \to \pi$ transitions (evaluated at zero momentum transfer), the decay constants f_K and f_π, the B_K parameter from neutral kaon mixing, and the kaon mixing matrix elements of new operators that arise in theories of physics beyond the Standard Model. Their implications for the CKM matrix elements V_{us} and V_{ud} were also discussed. Furthermore, results were reported for some of the low-energy constants of $SU(2)_L \times SU(2)_R$ and $SU(3)_L \times SU(3)_R$ Chiral Perturbation Theory. The quantities related to D- and B-meson physics that were reviewed were the masses of the charm and bottom quarks together with the decay constants, form factors, and mixing parameters of B- and D-mesons. These are the heavy-light quantities most relevant to the determination of CKM matrix elements and the global CKM unitarity-triangle fit. The current status of lattice results on the QCD coupling α_s was reviewed. Last but not least, we reviewed calculations of nucleon matrix elements of flavor nonsinglet and singlet bilinear operators, including the nucleon axial charge g_A and the nucleon sigma term. These results are relevant for constraining V_{ud}, for searches for new physics in neutron decays and other processes, and for dark matter searches.

In the present paper we provide updated results for all the above-mentioned quantities, but also extend the scope of the review by adding a section on scale setting, Sec. 11. The motivation for adding this section is that uncertainties in the value of the lattice spacing a

are a major source of error for the calculation of a wide range of quantities. Thus we felt that a systematic compilation of results, comparing the different approaches to setting the scale, and summarizing the present status, would be a useful resource for the lattice community. An additional update is the inclusion, in Sec. 6.2, of a brief description of the status of lattice calculations of $K \to \pi\pi$ decay amplitudes. Although some aspects of these calculations are not yet at the stage to be included in our averages, they are approaching this stage, and we felt that, given their phenomenological relevance, a brief review was appropriate.

For the most precisely determined quantities, isospin breaking—both from the up-down quark mass difference and from QED—must be included. A short review of methods used to include QED in lattice-QCD simulations is given in Sec. 3.1.3. An important issue here is that, in the context of a QED+QCD theory, the separation into QED and QCD contributions to a given physical quantity is ambiguous—there are several ways of defining such a separation. This issue is discussed from different viewpoints in the section on quark masses—see Sec. 3.1.1—and that on scale setting—see Sec. 11. We stress, however, that the physical observable in QCD+QED is defined unambiguously. Any ambiguity only arises because we are trying to separate a well-defined, physical quantity into two unphysical parts that provide useful information for phenomenology.

Our main results are collected in Tabs. 1, 2, 3, 4 and 5. As is clear from the tables, for most quantities there are results from ensembles with different values for N_f. In most cases, there is reasonable agreement among results with $N_f = 2$, $2+1$, and $2+1+1$. As precision increases, we may some day be able to distinguish among the different values of N_f, in which case, presumably $2+1+1$ would be the most realistic. (If isospin violation is critical, then $1+1+1$ or $1+1+1+1$ might be desired.) At present, for some quantities the errors in the $N_f = 2+1$ results are smaller than those with $N_f = 2+1+1$ (e.g., for m_c), while for others the relative size of the errors is reversed. Our suggestion to those using the averages is to take whichever of the $N_f = 2+1$ or $N_f = 2+1+1$ results has the smaller error. We do not recommend using the $N_f = 2$ results, except for studies of the N_f-dependence of condensates and α_s, as these have an uncontrolled systematic error coming from quenching the strange quark.

Our plan is to continue providing FLAG updates, in the form of a peer reviewed paper, roughly on a triennial basis. This effort is supplemented by our more frequently updated website `http://flag.unibe.ch` [5], where figures as well as pdf-files for the individual sections can be downloaded. The papers reviewed in the present edition have appeared before the closing date **30 April 2021**.[1]

[1] Working groups were given the option of including papers submitted to `arxiv.org` before the closing date but published after this date. This flexibility allows this review to be up-to-date at the time of submission. A single paper of this type was included.

Quantity	Sec.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f = 2$	Refs.
m_{ud}[MeV]	3.1.4	3.410(43)	[6, 7]	3.381(40)	[8–12]		
m_s[MeV]	3.1.4	93.40(57)	[6, 7, 13, 14]	92.2(1.0)	[8–11, 15]		
m_s/m_{ud}	3.1.5	27.23(10)	[7, 16, 17]	27.42(12)	[8–10, 15, 18]		
m_u[MeV]	3.1.6	2.14(8)	[6, 19]	2.27(9)	[20]		
m_d[MeV]	3.1.6	4.70(5)	[6, 19]	4.67(9)	[20]		
m_u/m_d	3.1.6	0.465(24)	[19, 21]	0.485(19)	[20]		
$\overline{m}_c$(3 GeV)[GeV]	3.2.2	0.988(11)	[6, 7, 14, 22, 23]	0.992(5)	[11, 24–26]		
m_c/m_s	3.2.3	11.768(34)	[6, 7, 14]	11.82(16)	[24, 27]		
$\overline{m}_b(\overline{m}_b)$[GeV]	3.3	4.203(11)	[6, 28–31]	4.171(20)	[11]		
$f_+(0)$	4.3	0.9698(17)	[32, 33]	0.9677(27)	[34, 35]	0.9560(57)(62)	[36]
$f_{K^\pm}/f_{\pi^\pm}$	4.3	1.1932(21)	[16, 37–39]	1.1917(37)	[8, 40–44]	1.205(18)	[45]
$f_{\pi^\pm}$[MeV]	4.6			130.2(8)	[8, 40, 41]		
$f_{K^\pm}$[MeV]	4.6	155.7(3)	[17, 37, 38]	155.7(7)	[8, 40, 41]	157.5(2.4)	[45]
Re(A_2)[GeV]	6.2			$1.50(4)(14) \times 10^{-8}$	[46]		
Im(A_2)[GeV]	6.2			$-8.34(1.03) \times 10^{-13}$	[46]		
$\hat{B}_K$	6.3	0.717(18)(16)	[47]	0.7625(97)	[8, 48–50]	0.727(22)(12)	[51]
B_2	6.4	0.46(1)(3)	[47]	0.502(14)	[50, 52]	0.47(2)(1)	[51]
B_3	6.4	0.79(2)(5)	[47]	0.766(32)	[50, 52]	0.78(4)(2)	[51]
B_4	6.4	0.78(2)(4)	[47]	0.926(19)	[50, 52]	0.76(2)(2)	[51]
B_5	6.4	0.49(3)(3)	[47]	0.720(38)	[50, 52]	0.58(2)(2)	[51]

Table 1: Summary of the main results of this review concerning quark masses, light-meson decay constants, and hadronic kaon-decay and kaon-mixing parameters. These are grouped in terms of N_f, the number of dynamical quark flavours in lattice simulations. Quark masses are given in the $\overline{\rm MS}$ scheme at running scale $\mu = 2\,$GeV or as indicated. BSM bag parameters $B_{2,3,4,5}$ are given in the $\overline{\rm MS}$ scheme at scale $\mu = 3\,$GeV. Further specifications of the quantities are given in the quoted sections. Results for $N_f = 2$ quark masses are unchanged since FLAG 16 [3], and are not included here. For each result we list the references that enter the FLAG average or estimate, and we stress again the importance of quoting these original works when referring to FLAG results. From the entries in this column one can also read off the number of results that enter our averages for each quantity. We emphasize that these numbers only give a very rough indication of how thoroughly the quantity in question has been explored on the lattice and recommend consulting the detailed tables and figures in the relevant section for more significant information and for explanations on the source of the quoted errors.

Quantity	Sec.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f = 2$	Refs.
f_D[MeV]	7.1	212.0(7)	[16, 38]	209.0(2.4)	[53–55]	208(7)	[56]
f_{D_s}[MeV]	7.1	249.9(5)	[16, 38]	248.0(1.6)	[24, 54, 55, 57]	246(4)	[56, 58]
$\frac{f_{D_s}}{f_D}$	7.1	1.1783(16)	[16, 38]	1.174(7)	[53–55]	1.20(2)	[56]
$f_+^{D\pi}(0)$	7.2	0.612(35)	[59]	0.666(29)	[60]		
$f_+^{DK}(0)$	7.2	0.7385(44)	[59, 61]	0.747(19)	[62]		
f_B[MeV]	8.1	190.0(1.3)	[16, 30, 63, 64]	192.0(4.3)	[54, 65–68]	188(7)	[56, 69]
f_{B_s}[MeV]	8.1	230.3(1.3)	[16, 30, 63, 64]	228.4(3.7)	[54, 65–68]	225.3(6.6)	[56, 58, 69]
$\frac{f_{B_s}}{f_B}$	8.1	1.209(5)	[16, 30, 63, 64]	1.201(16)	[54, 65–68, 70]	1.206(23)	[56, 69]
$f_{B_d}\sqrt{\hat{B}_{b_d}}$[MeV]	8.2	210.6(5.5)	[71]	225(9)	[67, 72, 73]	216(10)	[56]
$f_{B_s}\sqrt{\hat{B}_{B_s}}$[MeV]	8.2	256.1(5.7)	[71]	274(8)	[67, 72, 73]	262(10)	[56]
$\hat{B}_{B_d}$	8.2	1.222(61)	[71]	1.30(10)	[67, 72, 73]	1.30(6)	[56]
$\hat{B}_{B_s}$	8.2	1.232(53)	[71]	1.35(6)	[67, 72, 73]	1.32(5)	[56]
ξ	8.2	1.216(16)	[71]	1.206(17)	[67, 73]	1.225(31)	[56]
B_{B_s}/B_{B_d}	8.2	1.008(25)	[71]	1.032(38)	[67, 73]	1.007(21)	[56]
Quantity	Sec.	$N_f = 2+1$ and $N_f = 2+1+1$			Refs.		
$\alpha_{\overline{\rm MS}}^{(5)}(M_Z)$	9.11	0.1184(8)			[11, 14, 74–79]		
$\Lambda_{\overline{\rm MS}}^{(5)}$[MeV]	9.11	214(10)			[11, 14, 74–79]		
$\Lambda_{\overline{\rm MS}}^{(4)}$[MeV]	9.11	297(12)			[11, 14, 74–79]		
$\Lambda_{\overline{\rm MS}}^{(3)}$[MeV]	9.11	339(12)			[11, 14, 74–79]		

Table 2: Summary of the main results of this review concerning heavy-light mesons and the strong coupling constant. These are grouped in terms of N_f, the number of dynamical quark flavours in lattice simulations. The quantities listed are specified in the quoted sections. For each result we list the references that enter the FLAG average or estimate, and we stress again the importance of quoting these original works when referring to FLAG results. From the entries in this column one can also read off the number of results that enter our averages for each quantity. We emphasize that these numbers only give a very rough indication of how thoroughly the quantity in question has been explored on the lattice and recommend consulting the detailed tables and figures in the relevant section for more significant information and for explanations on the source of the quoted errors.

Quantity	Sec.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f = 2$	Refs.
$\Sigma^{1/3}$[MeV]	5.2.4	286(23)	[80, 81]	272(5)	[12, 82–86]	266(10)	[80, 87–89]
F_π/F	5.2.4	1.077(3)	[90]	1.062(7)	[41, 82–84, 91]	1.073(15)	[87–89, 92]
$\bar{\ell}_3$	5.2.4	3.53(26)	[90]	3.07(64)	[41, 82–84, 91]	3.41(82)	[87, 88, 92]
$\bar{\ell}_4$	5.2.4	4.73(10)	[90]	4.02(45)	[41, 82–84, 91]	4.40(28)	[87, 88, 92, 93]
$\bar{\ell}_6$	5.2.4					15.1(1.2)	[88, 92]
$a_0^2 M_\pi$	5.2.4	$-0.0441(4)$	[94]			$-0.04385(47)$	[95]
$\Sigma_0^{1/3}$[MeV]	5.3.5			245(8)	[15]		
Σ/Σ_0	5.3.5			1.48(16)	[15]		
F_0[MeV]	5.3.5			80.3(6.0)	[41]		
F/F_0	5.3.5			1.104(41)	[15]		
B/B_0	5.3.5			1.21(7)	[15]		
L_4	5.3.5	$+0.09(34) \times 10^{-3}$	[37]	$-0.02(56) \times 10^{-3}$	[41]		
L_5	5.3.5	$+1.19(25) \times 10^{-3}$	[37]	$+0.95(41) \times 10^{-3}$	[41]		
L_6	5.3.5	$+0.16(20) \times 10^{-3}$	[37]	$+0.01(34) \times 10^{-3}$	[41]		
L_8	5.3.5	$+0.55(15) \times 10^{-3}$	[37]	$+0.43(28) \times 10^{-3}$	[41]		
$a_0^{1/2}\mu_{\pi K}$	5.3.5	0.127(2)	[96]				
$a_0^{3/2}\mu_{\pi K}$	5.3.5	$-0.0463(17)$	[96]				
$a_0^1 M_K$	5.3.5	$-0.388(20)$	[97]				

Table 3: Summary of the main results of this review concerning LECs, grouped in terms of N_f, the number of dynamical quark flavours in lattice simulations. The quantities listed are specified in the quoted sections. For each result we list the references that enter the FLAG average or estimate, and we stress again the importance of quoting these original works when referring to FLAG results. From the entries in this column one can also read off the number of results that enter our averages for each quantity. We emphasize that these numbers only give a very rough indication of how thoroughly the quantity in question has been explored on the lattice and recommend consulting the detailed tables and figures in the relevant section for more significant information and for explanations on the source of the quoted errors.

Quantity	Sec.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f = 2$	Refs.
g_A^{u-d}	10.3.1	1.246(28)	[98–100]	1.248(23)	[101, 102]		
g_S^{u-d}	10.3.2	1.02(10)	[98]	1.13(14)	[102]		
g_T^{u-d}	10.3.3	0.989(34)	[98]	0.965(61)	[102]		
g_A^u	10.4.1	0.777(25)(30)	[103]	0.847(18)(32)	[101]		
g_A^d	10.4.1	−0.438(18)(30)	[103]	−0.407(16)(18)	[101]		
g_A^s	10.4.1	−0.053(8)	[103]	−0.035(6)(7)	[101]		
$\sigma_{\pi N}$[MeV]	10.4.4	64.9(1.5)(13.2)	[22]	39.7(3.6)	[104–106]	37(8)(6)	[107]
σ_s[MeV]	10.4.4	41.0(8.8)	[108]	52.9(7.0)	[104–106, 108, 109]		
g_T^u	10.4.5	0.784(28)(10)	[110]				
g_T^d	10.4.5	−0.204(11)(10)	[110]				
g_T^s	10.4.5	−0.0027(16)	[110]				

Table 4: Summary of the main results of this review concerning nuclear matrix elements, grouped in terms of N_f, the number of dynamical quark flavours in lattice simulations. The quantities listed are specified in the quoted sections. For each result we list the references that enter the FLAG average or estimate, and we stress again the importance of quoting these original works when referring to FLAG results. From the entries in this column one can also read off the number of results that enter our averages for each quantity. We emphasize that these numbers only give a very rough indication of how thoroughly the quantity in question has been explored on the lattice and recommend consulting the detailed tables and figures in the relevant section for more significant information and for explanations on the source of the quoted errors.

Quantity	Sec.	$N_f = 1+1+1+1$	Refs.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f > 2+1$	Refs.
$\sqrt{t_0}$[fm]	11.5.2			0.14186(88)	[37, 111, 112]	0.14464(87)	[8, 113, 114]		
w_0[fm]	11.5.2	0.17236(70)	[115]	0.17128(107)	[37, 111, 112]	0.17355(92)	[8, 114, 116]	0.17177(67)	[37, 111, 112, 115]
r_0[fm]	11.5.2			0.474(14)	[7]	0.4701(36)	[24, 116–119]		
r_1[fm]	11.5.2			0.3112(30)	[37]	0.3127(30)	[41, 117–120]		

Table 5: Summary of the main results of this review concerning setting of the lattice scale, grouped in terms of N_f, the number of dynamical quark flavours in lattice simulations. The quantities listed are specified in the quoted sections. For each result we list the references that enter the FLAG average or estimate, and we stress again the importance of quoting these original works when referring to FLAG results. From the entries in this column one can also read off the number of results that enter our averages for each quantity. We emphasize that these numbers only give a very rough indication of how thoroughly the quantity in question has been explored on the lattice and recommend consulting the detailed tables and figures in the relevant section for more significant information and for explanations on the source of the quoted errors.

This review is organized as follows. In the remainder of Sec. 1 we summarize the composition and rules of FLAG and discuss general issues that arise in modern lattice calculations. In Sec. 2, we explain our general methodology for evaluating the robustness of lattice results. We also describe the procedures followed for combining results from different collaborations in a single average or estimate (see Sec. 2.2 for our definition of these terms). The rest of the paper consists of sections, each dedicated to a set of closely connected physical quantities, or, for the final section, to the determination of the lattice scale. Each of these sections is accompanied by an Appendix with explicatory notes.[2]

In previous editions, we have provided, in an appendix, a glossary summarizing some standard lattice terminology and describing the most commonly used lattice techniques and methodologies. Since no significant updates in this information have occurred since our previous edition, we have decided, in the interests of reducing the length of this review, to omit this glossary, and refer the reader to FLAG 19 for this information [4]. This appendix also contained, in previous versions, a tabulation of the actions used in the papers that were reviewed. Since this information is available in the discussions in the separate sections, and is time-consuming to collect from the sections, we have dropped these tables. We have, however, kept a short appendix, Appendix A.1, describing the parameterizations of semileptonic form factors that are used in Sec. 8.

1.1 FLAG composition, guidelines and rules

FLAG strives to be representative of the lattice community, both in terms of the geographical location of its members and the lattice collaborations to which they belong. We aspire to provide the nuclear- and particle-physics communities with a single source of reliable information on lattice results.

In order to work reliably and efficiently, we have adopted a formal structure and a set of rules by which all FLAG members abide. The collaboration presently consists of an Advisory Board (AB), an Editorial Board (EB), and nine Working Groups (WG). The rôle of the Advisory Board is to provide oversight of the content, procedures, schedule and membership of FLAG, to help resolve disputes, to serve as a source of advice to the EB and to FLAG as a whole, and to provide a critical assessment of drafts. They also give their approval of the final version of the preprint before it is rendered public. The Editorial Board coordinates the activities of FLAG, sets priorities and intermediate deadlines, organizes votes on FLAG procedures, writes the introductory sections, and takes care of the editorial work needed to amalgamate the sections written by the individual working groups into a uniform and coherent review. The working groups concentrate on writing the review of the physical quantities for which they are responsible, which is subsequently circulated to the whole collaboration for critical evaluation.

The current list of FLAG members and their Working Group assignments is:

- Advisory Board (AB): G. Colangelo, M. Golterman, P. Hernandez, T. Onogi, and R. Van de Water
- Editorial Board (EB): S. Gottlieb, A. Jüttner, S. Hashimoto, S.R. Sharpe, and U. Wenger

[2] In some cases, in order to keep the length of this review within reasonable bounds, we have dropped these notes for older data, since they can be found in previous FLAG reviews [1–4] .

- Working Groups (coordinator listed first):
  - Quark masses — T. Blum, A. Portelli, and A. Ramos
  - V_{us}, V_{ud} — T. Kaneko, J. N. Simone, S. Simula, and N. Tantalo
  - LEC — S. Dürr, H. Fukaya, and U.M. Heller
  - B_K — P. Dimopoulos, X. Feng, and G. Herdoiza
  - $f_{B_{(s)}}$, $f_{D_{(s)}}$, B_B — Y. Aoki, M. Della Morte, and C. Monahan
  - b and c semileptonic and radiative decays — E. Lunghi, S. Meinel, and C. Pena
  - α_s — S. Sint, R. Horsley, and P. Petreczky
  - NME — R. Gupta, S. Collins, A. Nicholson, and H. Wittig
  - Scale setting — R. Sommer, N. Tantalo, and U. Wenger

The most important FLAG guidelines and rules are the following:

- the composition of the AB reflects the main geographical areas in which lattice collaborations are active, with members from America, Asia/Oceania, and Europe;
- the mandate of regular members is not limited in time, but we expect that a certain turnover will occur naturally;
- whenever a replacement becomes necessary this has to keep, and possibly improve, the balance in FLAG, so that different collaborations, from different geographical areas are represented;
- in all working groups the members must belong to different lattice collaborations;
- a paper is in general not reviewed (nor colour-coded, as described in the next section) by any of its authors;
- lattice collaborations will be consulted on the colour coding of their calculation;
- there are also internal rules regulating our work, such as voting procedures.

As for FLAG 19, for this review we sought the advice of external reviewers once a complete draft of the review was available. For each review section, we have asked one lattice expert (who could be a FLAG alumnus/alumna) and one nonlattice phenomenologist for a critical assessment. The one exception is the scale-setting section, where only a lattice expert has been asked to provide input. This is similar to the procedure followed by the Particle Data Group in the creation of the Review of Particle Physics. The reviewers provide comments and feedback on scientific and stylistic matters. They are not anonymous, and enter into a discussion with the authors of the WG. Our aim with this additional step is to make sure that a wider array of viewpoints enter into the discussions, so as to make this review more useful for its intended audience.

1.2 Citation policy

We draw attention to this particularly important point. As stated above, our aim is to make lattice-QCD results easily accessible to those without lattice expertise, and we are well aware that it is likely that some readers will only consult the present paper and not the original lattice literature. It is very important that this paper not be the only one cited when our results are quoted. We strongly suggest that readers also cite the original sources. In order to facilitate this, in Tabs. 1, 2, 3, 4, and 5, besides summarizing the main results of the present review, we also cite the original references from which they have been obtained. In addition, for each figure we make a bibtex file available on our webpage [5] which contains the bibtex entries of all the calculations contributing to the FLAG average or estimate. The bibliography at the end of this paper should also make it easy to cite additional papers. Indeed, we hope that the bibliography will be one of the most widely used elements of the whole paper.

1.3 General issues

Several general issues concerning the present review are thoroughly discussed in Sec. 1.1 of our initial 2010 paper [1], and we encourage the reader to consult the relevant pages. In the remainder of the present subsection, we focus on a few important points. Though the discussion has been duly updated, it is similar to that of Sec. 1.2 in the previous three reviews [2–4].

The present review aims to achieve two distinct goals: first, to provide a **description** of the relevant work done on the lattice; and, second, to draw **conclusions** on the basis of that work, summarizing the results obtained for the various quantities of physical interest.

The core of the information about the work done on the lattice is presented in the form of tables, which not only list the various results, but also describe the quality of the data that underlie them. We consider it important that this part of the review represents a generally accepted description of the work done. For this reason, we explicitly specify the quality requirements used and provide sufficient details in appendices so that the reader can verify the information given in the tables.[3]

On the other hand, the conclusions drawn on the basis of the available lattice results are the responsibility of FLAG alone. Preferring to err on the side of caution, in several cases we draw conclusions that are more conservative than those resulting from a plain weighted average of the available lattice results. This cautious approach is usually adopted when the average is dominated by a single lattice result, or when only one lattice result is available for a given quantity. In such cases, one does not have the same degree of confidence in results and errors as when there is agreement among several different calculations using different approaches. The reader should keep in mind that the degree of confidence cannot be quantified, and it is not reflected in the quoted errors.

Each discretization has its merits, but also its shortcomings. For most topics covered in this review we have an increasingly broad database, and for most quantities lattice calculations based on totally different discretizations are now available. This is illustrated by the dense population of the tables and figures in most parts of this review. Those calculations that do satisfy our quality criteria indeed lead, in almost all cases, to consistent results, confirming universality within the accuracy reached. The consistency between independent

[3] We also use terms like "quality criteria", "rating", "colour coding", etc., when referring to the classification of results, as described in Sec. 2.

lattice results, obtained with different discretizations, methods, and simulation parameters, is an important test of lattice QCD, and observing such consistency also provides further evidence that systematic errors are fully under control.

In the sections dealing with heavy quarks and with α_s, the situation is not the same. Since the b-quark mass can barely be resolved with current lattice spacings, most lattice methods for treating b quarks use effective field theory at some level. This introduces additional complications not present in the light-quark sector. An overview of the issues specific to heavy-quark quantities is given in the introduction of Sec. 8. For B- and D-meson leptonic decay constants, there already exists a good number of different independent calculations that use different heavy-quark methods, but there are only a few independent calculations of semileptonic B, Λ_b, and D form factors and of $B-\bar{B}$ mixing parameters. For α_s, most lattice methods involve a range of scales that need to be resolved and controlling the systematic error over a large range of scales is more demanding. The issues specific to determinations of the strong coupling are summarized in Sec. 9.

Number of sea quarks in lattice simulations:
Lattice-QCD simulations currently involve two, three or four flavours of dynamical quarks. Most simulations set the masses of the two lightest quarks to be equal, while the strange and charm quarks, if present, are heavier (and tuned to lie close to their respective physical values). Our notation for these simulations indicates which quarks are nondegenerate, e.g., $N_f = 2+1$ if $m_u = m_d < m_s$ and $N_f = 2+1+1$ if $m_u = m_d < m_s < m_c$. Calculations with $N_f = 2$, i.e., two degenerate dynamical flavours, often include strange valence quarks interacting with gluons, so that bound states with the quantum numbers of the kaons can be studied, albeit neglecting strange sea-quark fluctuations. The quenched approximation ($N_f = 0$), in which all sea-quark contributions are omitted, has uncontrolled systematic errors and is no longer used in modern lattice simulations with relevance to phenomenology. Accordingly, we will review results obtained with $N_f = 2$, $N_f = 2+1$, and $N_f = 2+1+1$, but omit earlier results with $N_f = 0$. The only exception concerns the QCD coupling constant α_s. Since this observable does not require valence light quarks, it is theoretically well defined also in the $N_f = 0$ theory, which is simply pure gluodynamics. The N_f-dependence of α_s, or more precisely of the related quantity $r_0\Lambda_{\overline{\rm MS}}$, is a theoretical issue of considerable interest; here r_0 is a quantity with the dimension of length that sets the physical scale, as discussed in Sec. 11. We stress, however, that only results with $N_f \geq 3$ are used to determine the physical value of α_s at a high scale.

Lattice actions, simulation parameters, and scale setting:
The remarkable progress in the precision of lattice calculations is due to improved algorithms, better computing resources, and, last but not least, conceptual developments. Examples of the latter are improved actions that reduce lattice artifacts and actions that preserve chiral symmetry to very good approximation. A concise characterization of the various discretizations that underlie the results reported in the present review is given in Appendix A.1 of FLAG 19.

Physical quantities are computed in lattice simulations in units of the lattice spacing so that they are dimensionless. For example, the pion decay constant that is obtained from a simulation is $f_\pi a$, where a is the spacing between two neighboring lattice sites. (All simulations with results quoted in this review use hypercubic lattices, i.e., with the same spacing in all four Euclidean directions.) To convert these results to physical units requires knowledge of the lattice spacing a at the fixed values of the bare QCD parameters (quark masses and gauge coupling) used in the simulation. This is achieved by requiring agreement between the lattice calculation and experimental measurement of a known quantity, which thus "sets the scale" of a given simulation. Given the central importance of this procedure, we include in this edition of FLAG a dedicated section, Sec. 11, discussing the issues and results.

Renormalization and scheme dependence:
Several of the results covered by this review, such as quark masses, the gauge coupling, and B-parameters, are for quantities defined in a given renormalization scheme and at a specific renormalization scale. The schemes employed (e.g., regularization-independent MOM schemes) are often chosen because of their specific merits when combined with the lattice regularization. For a brief discussion of their properties, see Appendix A.3 of FLAG 19. The conversion of the results obtained in these so-called intermediate schemes to more familiar regularization schemes, such as the $\overline{\mathrm{MS}}$-scheme, is done with the aid of perturbation theory. It must be stressed that the renormalization scales accessible in simulations are limited, because of the presence of an ultraviolet (UV) cutoff of $\sim \pi/a$. To safely match to $\overline{\mathrm{MS}}$, a scheme defined in perturbation theory, Renormalization Group (RG) running to higher scales is performed, either perturbatively or nonperturbatively (the latter using finite-size scaling techniques).

Extrapolations:
Because of limited computing resources, lattice simulations are often performed at unphysically heavy pion masses, although results at the physical point have become increasingly common. Further, numerical simulations must be done at nonzero lattice spacing, and in a finite (four-dimensional) volume. In order to obtain physical results, lattice data are obtained at a sequence of pion masses and a sequence of lattice spacings, and then extrapolated to the physical pion mass and to the continuum limit. In principle, an extrapolation to infinite volume is also required. However, for most quantities discussed in this review, finite-volume effects are exponentially small in the linear extent of the lattice in units of the pion mass, and, in practice, one often verifies volume independence by comparing results obtained on a few different physical volumes, holding other parameters fixed. To control the associated systematic uncertainties, these extrapolations are guided by effective theories. For light-quark actions, the lattice-spacing dependence is described by Symanzik's effective theory [121, 122]; for heavy quarks, this can be extended and/or supplemented by other effective theories such as Heavy-Quark Effective Theory (HQET). The pion-mass dependence can be parameterized with Chiral Perturbation Theory (χPT), which takes into account the Nambu-Goldstone nature of the lowest excitations that occur in the presence of light quarks. Similarly, one can use Heavy-Light Meson Chiral Perturbation Theory (HMχPT) to extrapolate quantities involving mesons composed of one heavy (b or c) and one light quark. One can combine Symanzik's effective theory with χPT to simultaneously extrapolate to the physical pion mass and the continuum; in this case, the form of the effective theory depends on the discretization. See Appendix A.4 of FLAG 19 for a brief description of the different variants in use and some useful references. Finally, χPT can also be used to estimate the size of finite-volume effects measured in units of the inverse pion mass, thus providing information on the systematic error due to finite-volume effects in addition to that obtained by comparing simulations at different volumes.

Excited-state contamination:
In all the hadronic matrix elements discussed in this review, the hadron in question is the lightest state with the chosen quantum numbers. This implies that it dominates the required

correlation functions as their extent in Euclidean time is increased. Excited-state contributions are suppressed by $e^{-\Delta E \Delta \tau}$, where ΔE is the gap between the ground and excited states, and $\Delta\tau$ the relevant separation in Euclidean time. The size of ΔE depends on the hadron in question, and in general is a multiple of the pion mass. In practice, as discussed at length in Sec. 10, the contamination of signals due to excited-state contributions is a much more challenging problem for baryons than for the other particles discussed here. This is in part due to the fact that the signal-to-noise ratio drops exponentially for baryons, which reduces the values of $\Delta\tau$ that can be used.

Critical slowing down:
The lattice spacings reached in recent simulations go down to 0.05 fm or even smaller. In this regime, long autocorrelation times slow down the sampling of the configurations [123–132]. Many groups check for autocorrelations in a number of observables, including the topological charge, for which a rapid growth of the autocorrelation time is observed with decreasing lattice spacing. This is often referred to as topological freezing. A solution to the problem consists in using open boundary conditions in time [133], instead of the more common antiperiodic ones. More recently, two other approaches have been proposed, one based on a multiscale thermalization algorithm [134, 135] and another based on defining QCD on a nonorientable manifold [136]. The problem is also touched upon in Sec. 9.2.1, where it is stressed that attention must be paid to this issue. While large scale simulations with open boundary conditions are already far advanced [137], few results reviewed here have been obtained with any of the above methods. It is usually *assumed* that the continuum limit can be reached by extrapolation from the existing simulations, and that potential systematic errors due to the long autocorrelation times have been adequately controlled. Partially or completely frozen topology would produce a mixture of different θ vacua, and the difference from the desired $\theta = 0$ result may be estimated in some cases using chiral perturbation theory, which gives predictions for the θ-dependence of the physical quantity of interest [138, 139]. These ideas have been systematically and successfully tested in various models in [140, 141], and a numerical test on MILC ensembles indicates that the topology dependence for some of the physical quantities reviewed here is small, consistent with theoretical expectations [142].

Simulation algorithms and numerical errors:
Most of the modern lattice-QCD simulations use exact algorithms such as those of Refs. [143, 144], which do not produce any systematic errors when exact arithmetic is available. In reality, one uses numerical calculations at double (or in some cases even single) precision, and some errors are unavoidable. More importantly, the inversion of the Dirac operator is carried out iteratively and it is truncated once some accuracy is reached, which is another source of potential systematic error. In most cases, these errors have been confirmed to be much less than the statistical errors. In the following we assume that this source of error is negligible. Some of the most recent simulations use an inexact algorithm in order to speed up the computation, though it may produce systematic effects. Currently available tests indicate that errors from the use of inexact algorithms are under control [145].

References

[1] [FLAG 10] G. Colangelo, S. Dürr, A. Jüttner, L. Lellouch, H. Leutwyler et al., *Review of lattice results concerning low energy particle physics*, *Eur.Phys.J.* **C71** (2011) 1695 [1011.4408].

[2] [FLAG 13] S. Aoki, Y. Aoki, C. Bernard, T. Blum, G. Colangelo et al., *Review of lattice results concerning low-energy particle physics*, *Eur.Phys.J.* **C74** (2014) 2890 [1310.8555].

[3] [FLAG 16] S. Aoki et al., *Review of lattice results concerning low-energy particle physics*, *Eur. Phys. J.* **C77** (2017) 112 [1607.00299].

[4] [FLAG 19] S. Aoki et al., *FLAG Review 2019: Flavour Lattice Averaging Group (FLAG)*, *Eur. Phys. J. C* **80** (2020) 113 [1902.08191].

[5] Flavour Lattice Averaging Group (FLAG), *FLAG Review*, http://flag.unibe.ch/.

[6] [FNAL/MILC/TUMQCD 18] A. Bazavov et al., *Up-, down-, strange-, charm-, and bottom-quark masses from four-flavor lattice QCD*, *Phys. Rev.* **D98** (2018) 054517 [1802.04248].

[7] [ETM 14] N. Carrasco et al., *Up, down, strange and charm quark masses with N_f = 2+1+1 twisted mass lattice QCD*, *Nucl. Phys.* **B887** (2014) 19 [1403.4504].

[8] [RBC/UKQCD 14B] T. Blum et al., *Domain wall QCD with physical quark masses*, *Phys. Rev.* **D93** (2016) 074505 [1411.7017].

[9] [BMW 10A] S. Dürr, Z. Fodor, C. Hoelbling, S. Katz, S. Krieg et al., *Lattice QCD at the physical point: light quark masses*, *Phys.Lett.* **B701** (2011) 265 [1011.2403].

[10] [BMW 10B] S. Dürr, Z. Fodor, C. Hoelbling, S. Katz, S. Krieg et al., *Lattice QCD at the physical point: simulation and analysis details*, *JHEP* **1108** (2011) 148 [1011.2711].

[11] [HPQCD 10] C. McNeile, C. T. H. Davies, E. Follana, K. Hornbostel and G. P. Lepage, *High-precision c and b masses and QCD coupling from current-current correlators in lattice and continuum QCD*, *Phys. Rev.* **D82** (2010) 034512 [1004.4285].

[12] [MILC 10A] A. Bazavov et al., *Staggered chiral perturbation theory in the two-flavor case and SU(2) analysis of the MILC data*, *PoS* **LAT2010** (2010) 083 [1011.1792].

[13] [HPQCD 18] A. T. Lytle, C.T.H. Davies, D. Hatton, G.P. Lepage and C. Sturm, *Determination of quark masses from $\mathbf{n_f = 4}$ lattice QCD and the RI-SMOM intermediate scheme*, *Phys. Rev.* **D98** (2018) 014513 [1805.06225].

[14] [HPQCD 14A] B. Chakraborty, C.T.H. Davies, G.C. Donald, R.J. Dowdall, B. Galloway, P. Knecht et al., *High-precision quark masses and QCD coupling from $n_f = 4$ lattice QCD*, *Phys.Rev.* **D91** (2015) 054508 [1408.4169].

[15] [MILC 09A] A. Bazavov et al., *MILC results for light pseudoscalars*, *PoS* **CD09** (2009) 007 [0910.2966].

[16] [FNAL/MILC 17] A. Bazavov et al., *B- and D-meson leptonic decay constants from four-flavor lattice QCD*, *Phys. Rev.* **D98** (2018) 074512 [1712.09262].

[17] [FNAL/MILC 14A] A. Bazavov et al., *Charmed and light pseudoscalar meson decay constants from four-flavor lattice QCD with physical light quarks*, *Phys.Rev.* **D90** (2014) 074509 [1407.3772].

[18] [ALPHA 19] M. Bruno, I. Campos, P. Fritzsch, J. Koponen, C. Pena, D. Preti et al., *Light quark masses in $N_{\rm f} = 2+1$ lattice QCD with Wilson fermions*, *Eur. Phys. J. C* **80** (2020) 169 [1911.08025].

[19] [RM123 17] D. Giusti, V. Lubicz, C. Tarantino, G. Martinelli, S. Sanfilippo, S. Simula et al., *Leading isospin-breaking corrections to pion, kaon and charmed-meson masses with Twisted-Mass fermions*, *Phys. Rev.* **D95** (2017) 114504 [1704.06561].

[20] [BMW 16] Z. Fodor, C. Hoelbling, S. Krieg, L. Lellouch, T. Lippert, A. Portelli et al., *Up and down quark masses and corrections to Dashen's theorem from lattice QCD and quenched QED*, *Phys. Rev. Lett.* **117** (2016) 082001 [1604.07112].

[21] [MILC 18] S. Basak et al., *Lattice computation of the electromagnetic contributions to kaon and pion masses*, *Phys. Rev.* **D99** (2019) 034503 [1807.05556].

[22] [ETM 14A] C. Alexandrou, V. Drach, K. Jansen, C. Kallidonis and G. Koutsou, *Baryon spectrum with $N_f = 2+1+1$ twisted mass fermions*, *Phys. Rev.* **D90** (2014) 074501 [1406.4310].

[23] [HPQCD 20A] D. Hatton, C.T.H. Davies, B. Galloway, J. Koponen, G.P. Lepage and A.T. Lytle, *Charmonium properties from lattice QCD+QED : Hyperfine splitting, J/ψ leptonic width, charm quark mass, and a_μ^c*, *Phys. Rev. D* **102** (2020) 054511 [2005.01845].

[24] [χQCD 14] Y. Yi-Bo et al., *Charm and strange quark masses and f_D_s from overlap fermions*, *Phys. Rev.* **D92** (2015) 034517 [1410.3343].

[25] [JLQCD 16] K. Nakayama, B. Fahy and S. Hashimoto, *Short-distance charmonium correlator on the lattice with Möbius domain-wall fermion and a determination of charm quark mass*, *Phys. Rev.* **D94** (2016) 054507 [1606.01002].

[26] P. Petreczky and J. Weber, *Strong coupling constant and heavy quark masses in (2+1)-flavor QCD*, *Phys. Rev. D* **100** (2019) 034519 [1901.06424].

[27] [HPQCD 09A] C. T. H. Davies et al., *Precise charm to strange mass ratio and light quark masses from full lattice QCD*, *Phys. Rev. Lett.* **104** (2010) 132003 [0910.3102].

[28] [HPQCD 21] D. Hatton, C.T.H. Davies, J. Koponen, G.P. Lepage and A.T. Lytle, *Determination of $\overline{m}_b/\overline{m}_c$ and $\overline{m}_b$ from $n_f = 4$ lattice QCD+QED*, *Phys. Rev. D* **103** (2021) 114508 [2102.09609].

[29] [HPQCD 14B] B. Colquhoun, R.J. Dowdall, C.T.H. Davies, K. Hornbostel and G.P. Lepage, *Υ and Υ' Leptonic Widths, a_μ^b and m_b from full lattice QCD*, *Phys. Rev.* **D91** (2015) 074514 [1408.5768].

[30] [ETM 16B] A. Bussone et al., *Mass of the b quark and B -meson decay constants from N_f=2+1+1 twisted-mass lattice QCD*, *Phys. Rev.* **D93** (2016) 114505 [1603.04306].

[31] P. Gambino, A. Melis and S. Simula, *Extraction of heavy-quark-expansion parameters from unquenched lattice data on pseudoscalar and vector heavy-light meson masses*, *Phys. Rev.* **D96** (2017) 014511 [1704.06105].

[32] [ETM 16] N. Carrasco, P. Lami, V. Lubicz, L. Riggio, S. Simula and C. Tarantino, *$K \to \pi$ semileptonic form factors with $N_f = 2+1+1$ twisted mass fermions*, *Phys. Rev.* **D93** (2016) 114512 [1602.04113].

[33] [FNAL/MILC 18] A. Bazavov et al., *$|V_{us}|$ from $K_{\ell 3}$ decay and four-flavor lattice QCD*, *Phys. Rev.* **D99** (2019) 114509 [1809.02827].

[34] [FNAL/MILC 12I] A. Bazavov, C. Bernard, C. Bouchard, C. DeTar, D. Du et al., *Kaon semileptonic vector form factor and determination of $|V_{us}|$ using staggered fermions*, *Phys.Rev.* **D87** (2013) 073012 [1212.4993].

[35] [RBC/UKQCD 15A] P.A. Boyle et al., *The kaon semileptonic form factor in $N_f = 2 + 1$ domain wall lattice QCD with physical light quark masses*, *JHEP* **1506** (2015) 164 [1504.01692].

[36] [ETM 09A] V. Lubicz, F. Mescia, S. Simula and C. Tarantino, *$K \to \pi \ell \nu$ semileptonic form factors from two-flavor lattice QCD*, *Phys. Rev.* **D80** (2009) 111502 [0906.4728].

[37] [HPQCD 13A] R. Dowdall, C. Davies, G. Lepage and C. McNeile, *V_{us} from π and K decay constants in full lattice QCD with physical u, d, s and c quarks*, *Phys.Rev.* **D88** (2013) 074504 [1303.1670].

[38] [ETM 14E] N. Carrasco, P. Dimopoulos, R. Frezzotti, P. Lami, V. Lubicz et al., *Leptonic decay constants f_K, f_D and f_{D_s} with $N_f = 2+1+1$ twisted-mass lattice QCD*, *Phys.Rev.* **D91** (2015) 054507 [1411.7908].

[39] [CalLat 20] N. Miller et al., *f_k/f_π from Möbius domain-wall fermions solved on gradient-flowed hisq ensembles*, *Phys. Rev. D* **102** (2020) 034507 [2005.04795].

[40] [HPQCD/UKQCD 07] E. Follana, C.T.H. Davies, G.P. Lepage and J. Shigemitsu, *High precision determination of the π, K, D and D_s decay constants from lattice QCD*, *Phys. Rev. Lett.* **100** (2008) 062002 [0706.1726].

[41] [MILC 10] A. Bazavov et al., *Results for light pseudoscalar mesons*, *PoS* **LAT2010** (2010) 074 [1012.0868].

[42] [BMW 10] S. Dürr, Z. Fodor, C. Hoelbling, S. Katz, S. Krieg et al., *The ratio F_K/F_π in QCD*, *Phys.Rev.* **D81** (2010) 054507 [1001.4692].

[43] S. Dürr et al., *Leptonic decay-constant ratio f_K/f_π from lattice QCD using 2+1 clover-improved fermion flavors with 2-HEX smearing*, *Phys. Rev.* **D95** (2017) 054513 [1601.05998].

[44] [QCDSF/UKQCD 16] V. G. Bornyakov, R. Horsley, Y. Nakamura, H. Perlt, D. Pleiter, P.E.L. Rakow et al., *Flavour breaking effects in the pseudoscalar meson decay constants*, *Phys. Lett.* **B767** (2017) 366 [1612.04798].

[45] [ETM 09] B. Blossier et al., *Pseudoscalar decay constants of kaon and D-mesons from $N_f = 2$ twisted mass lattice QCD*, *JHEP* **0907** (2009) 043 [0904.0954].

[46] [RBC/UKQCD 15F] T. Blum et al., *$K \to \pi\pi$ $\Delta I = 3/2$ decay amplitude in the continuum limit*, *Phys. Rev.* **D91** (2015) 074502 [1502.00263].

[47] [ETM 15] N. Carrasco, P. Dimopoulos, R. Frezzotti, V. Lubicz, G.C. Rossi, S. Simula et al., *$\Delta S = 2$ and $\Delta C = 2$ bag parameters in the standard model and beyond from $N_f = 2+1+1$ twisted-mass lattice QCD*, *Phys. Rev.* **D92** (2015) 034516 [1505.06639].

[48] [BMW 11] S. Dürr, Z. Fodor, C. Hoelbling, S. Katz, S. Krieg et al., *Precision computation of the kaon bag parameter*, *Phys.Lett.* **B705** (2011) 477 [1106.3230].

[49] J. Laiho and R.S. Van de Water, *Pseudoscalar decay constants, light-quark masses and B_K from mixed-action lattice QCD*, *PoS* **LATTICE2011** (2011) 293 [1112.4861].

[50] [SWME 15A] Y.-C. Jang et al., *Kaon BSM B-parameters using improved staggered fermions from $N_f = 2+1$ unquenched QCD*, *Phys. Rev.* **D93** (2016) 014511 [1509.00592].

[51] [ETM 12D] V. Bertone et al., *Kaon Mixing Beyond the SM from N_f=2 tmQCD and model independent constraints from the UTA*, *JHEP* **03** (2013) 089 [1207.1287], [Erratum: JHEP07,143(2013)].

[52] [RBC/UKQCD 16] N. Garron, R.J. Hudspith and A.T. Lytle, *Neutral Kaon Mixing Beyond the Standard Model with $n_f = 2+1$ Chiral Fermions Part 1: Bare Matrix Elements and Physical Results*, *JHEP* **11** (2016) 001 [1609.03334].

[53] [HPQCD 12A] H. Na, C.T. Davies, E. Follana, G.P. Lepage and J. Shigemitsu, *$|V_{cd}|$ from D meson leptonic decays*, *Phys.Rev.* **D86** (2012) 054510 [1206.4936].

[54] [FNAL/MILC 11] A. Bazavov et al., *B- and D-meson decay constants from three-flavor lattice QCD*, *Phys.Rev.* **D85** (2012) 114506 [1112.3051].

[55] [RBC/UKQCD 17] P. A. Boyle, L. Del Debbio, A. Jttner, A. Khamseh, F. Sanfilippo and J.T. Tsang, *The decay constants $\mathbf{f_D}$ and $\mathbf{f_{D_s}}$ in the continuum limit of $\mathbf{N_f = 2+1}$ domain wall lattice QCD*, *JHEP* **12** (2017) 008 [1701.02644].

[56] [ETM 13B] N. Carrasco et al., *B-physics from $N_f = 2$ tmQCD: the Standard Model and beyond*, *JHEP* **1403** (2014) 016 [1308.1851].

[57] [HPQCD 10A] C. T. H. Davies, C. McNeile, E. Follana, G. Lepage, H. Na et al., *Update: precision D_s decay constant from full lattice QCD using very fine lattices*, *Phys.Rev.* **D82** (2010) 114504 [1008.4018].

[58] R. Balasubramamian and B. Blossier, *Decay constant of B_s and B_s^* mesons from* $\mathrm{N_f} = 2$ *lattice QCD*, *Eur. Phys. J. C* **80** (2020) 412 [1912.09937].

[59] [ETM 17D] V. Lubicz, L. Riggio, G. Salerno, S. Simula and C. Tarantino, *Scalar and vector form factors of $D \to \pi(K)\ell\nu$ decays with $N_f = 2+1+1$ twisted fermions*, *Phys. Rev.* **D96** (2017) 054514 [1706.03017].

[60] [HPQCD 11] H. Na et al., *$D \to \pi\ell\nu$ semileptonic decays, $|V_{cd}|$ and 2^{nd} row unitarity from lattice QCD*, *Phys.Rev.* **D84** (2011) 114505 [1109.1501].

[61] [HPQCD 21A] B. Chakraborty, W.G. Parrott, C. Bouchard, C.T.H. Davies, J. Koponen and G.P. Lepage, *Improved V_{cs} determination using precise lattice QCD form factors for $D \to K\ell\nu$*, *Phys. Rev. D* **104** (2021) 034505 [2104.09883].

[62] [HPQCD 10B] H. Na, C.T.H. Davies, E. Follana, G.P. Lepage and J. Shigemitsu, *The $D \to K\ell\nu$ semileptonic decay scalar form factor and $|V_{cs}|$ from lattice QCD*, *Phys.Rev.* **D82** (2010) 114506 [1008.4562].

[63] [HPQCD 13] R. J. Dowdall, C. Davies, R. Horgan, C. Monahan and J. Shigemitsu, *B-meson decay constants from improved lattice NRQCD and physical u, d, s and c sea quarks*, *Phys.Rev.Lett.* **110** (2013) 222003 [1302.2644].

[64] [HPQCD 17A] C. Hughes, C.T.H. Davies and C.J. Monahan, *New methods for B meson decay constants and form factors from lattice NRQCD*, *Phys. Rev.* **D97** (2018) 054509 [1711.09981].

[65] [HPQCD 11A] C. McNeile, C.T.H. Davies, E. Follana, K. Hornbostel and G.P. Lepage, *High-precision f_{B_s} and HQET from relativistic lattice QCD*, *Phys.Rev.* **D85** (2012) 031503 [1110.4510].

[66] [HPQCD 12] H. Na, C.J. Monahan, C.T. Davies, R. Horgan, G.P. Lepage et al., *The B and B_s meson decay constants from lattice QCD*, *Phys.Rev.* **D86** (2012) 034506 [1202.4914].

[67] [RBC/UKQCD 14A] Y. Aoki, T. Ishikawa, T. Izubuchi, C. Lehner and A. Soni, *Neutral B meson mixings and B meson decay constants with static heavy and domain-wall light quarks*, *Phys. Rev.* **D91** (2015) 114505 [1406.6192].

[68] [RBC/UKQCD 14] N. H. Christ, J.M. Flynn, T. Izubuchi, T. Kawanai, C. Lehner et al., *B-meson decay constants from 2+1-flavor lattice QCD with domain-wall light quarks and relativistic heavy quarks*, *Phys.Rev.* **D91** (2015) 054502 [1404.4670].

[69] [ALPHA 14] F. Bernardoni et al., *Decay constants of B-mesons from non-perturbative HQET with two light dynamical quarks*, *Phys.Lett.* **B735** (2014) 349 [1404.3590].

[70] [RBC/UKQCD 18A] P. A. Boyle, L. Del Debbio, N. Garron, A. Juttner, A. Soni, J.T. Tsang et al., *SU(3)-breaking ratios for $D_{(s)}$ and $B_{(s)}$ mesons*, 1812.08791.

[71] [HPQCD 19A] R. J. Dowdall, C.T.H. Davies, R.R. Horgan, G.P. Lepage, C.J. Monahan, J. Shigemitsu et al., *Neutral B-meson mixing from full lattice QCD at the physical point*, *Phys. Rev.* **D100** (2019) 094508 [1907.01025].

[72] [HPQCD 09] E. Gamiz, C.T. Davies, G.P. Lepage, J. Shigemitsu and M. Wingate, *Neutral B meson mixing in unquenched lattice QCD*, *Phys.Rev.* **D80** (2009) 014503 [0902.1815].

[73] [FNAL/MILC 16] A. Bazavov et al., *$B^0_{(s)}$-mixing matrix elements from lattice QCD for the Standard Model and beyond*, *Phys. Rev.* **D93** (2016) 113016 [1602.03560].

[74] C. Ayala, X. Lobregat and A. Pineda, *Determination of $\alpha(M_z)$ from an hyperasymptotic approximation to the energy of a static quark-antiquark pair*, *JHEP* **09** (2020) 016 [2005.12301].

[75] [TUMQCD 19] A. Bazavov, N. Brambilla, X. Garcia i Tormo, P. Petreczky, J. Soto, A. Vairo et al., *Determination of the QCD coupling from the static energy and the free energy*, *Phys. Rev. D* **100** (2019) 114511 [1907.11747].

[76] S. Cali, K. Cichy, P. Korcyl and J. Simeth, *Running coupling constant from position-space current-current correlation functions in three-flavor lattice QCD*, Phys. Rev. Lett. **125** (2020) 242002 [2003.05781].

[77] [ALPHA 17] M. Bruno, M. Dalla Brida, P. Fritzsch, T. Korzec, A. Ramos, S. Schaefer et al., *QCD Coupling from a Nonperturbative Determination of the Three-Flavor Λ Parameter*, Phys. Rev. Lett. **119** (2017) 102001 [1706.03821].

[78] [PACS-CS 09A] S. Aoki et al., *Precise determination of the strong coupling constant in $N_f = 2+1$ lattice QCD with the Schrödinger functional scheme*, JHEP **0910** (2009) 053 [0906.3906].

[79] K. Maltman, D. Leinweber, P. Moran and A. Sternbeck, *The realistic lattice determination of $\alpha_s(M_Z)$ revisited*, *Phys. Rev.* **D78** (2008) 114504 [0807.2020].

[80] [ETM 13] K. Cichy, E. Garcia-Ramos and K. Jansen, *Chiral condensate from the twisted mass Dirac operator spectrum*, JHEP **1310** (2013) 175 [1303.1954].

[81] [ETMC 17E] C. Alexandrou, A. Athenodorou, K. Cichy, M. Constantinou, D.P. Horkel, K. Jansen et al., *Topological susceptibility from twisted mass fermions using spectral projectors and the gradient flow*, Phys. Rev. **D97** (2018) 074503 [1709.06596].

[82] S. Borsanyi, S. Dürr, Z. Fodor, S. Krieg, A. Schäfer et al., *SU(2) chiral perturbation theory low-energy constants from 2+1 flavor staggered lattice simulations*, Phys.Rev. **D88** (2013) 014513 [1205.0788].

[83] [BMW 13] S. Dürr, Z. Fodor, C. Hoelbling, S. Krieg, T. Kurth et al., *Lattice QCD at the physical point meets SU(2) chiral perturbation theory*, Phys. Rev. **D90** (2014) 114504 [1310.3626].

[84] [RBC/UKQCD 15E] P. A. Boyle et al., *Low energy constants of SU(2) partially quenched chiral perturbation theory from N_f=2+1 domain wall QCD*, Phys. Rev. **D93** (2016) 054502 [1511.01950].

[85] [JLQCD 16B] G. Cossu, H. Fukaya, S. Hashimoto, T. Kaneko and J.-I. Noaki, *Stochastic calculation of the Dirac spectrum on the lattice and a determination of chiral condensate in 2+1-flavor QCD*, PTEP **2016** (2016) 093B06 [1607.01099].

[86] [JLQCD 17A] S. Aoki, G. Cossu, H. Fukaya, S. Hashimoto and T. Kaneko, *Topological susceptibility of QCD with dynamical Möbius domain wall fermions*, PTEP **2018** (2018) 043B07 [1705.10906].

[87] [ETM 09C] R. Baron et al., *Light meson physics from maximally twisted mass lattice QCD*, JHEP **08** (2010) 097 [0911.5061].

[88] B. B. Brandt, A. Jüttner and H. Wittig, *The pion vector form factor from lattice QCD and NNLO chiral perturbation theory*, JHEP **1311** (2013) 034 [1306.2916].

[89] G.P. Engel, L. Giusti, S. Lottini and R. Sommer, *Spectral density of the Dirac operator in two-flavor QCD*, Phys. Rev. **D91** (2015) 054505 [1411.6386].

[90] [ETM 11] R. Baron et al., *Light hadrons from $N_f = 2+1+1$ dynamical twisted mass fermions*, *PoS* **LAT2010** (2010) 123 [1101.0518].

[91] [NPLQCD 11] S. R. Beane, W. Detmold, P. Junnarkar, T. Luu, K. Orginos et al., *SU(2) low-energy constants from mixed-action lattice QCD*, *Phys.Rev.* **D86** (2012) 094509 [1108.1380].

[92] [ETM 08] R. Frezzotti, V. Lubicz and S. Simula, *Electromagnetic form factor of the pion from twisted-mass lattice QCD at $N_f = 2$*, *Phys. Rev.* **D79** (2009) 074506 [0812.4042].

[93] V. Gülpers, G. von Hippel and H. Wittig, *The scalar radius of the pion from lattice QCD in the continuum limit*, *Eur. Phys. J.* **A51** (2015) 158 [1507.01749].

[94] [ETM 15E] C. Helmes, C. Jost, B. Knippschild, C. Liu, J. Liu, L. Liu et al., *Hadron-hadron interactions from $N_f = 2+1+1$ lattice QCD: isospin-2 π-π scattering length*, *JHEP* **09** (2015) 109 [1506.00408].

[95] [ETM 09G] X. Feng, K. Jansen and D.B. Renner, *The pi+ pi+ scattering length from maximally twisted mass lattice QCD*, *Phys. Lett.* **B684** (2010) 268 [0909.3255].

[96] [ETM 18B] C. Helmes, C. Jost, B. Knippschild, B. Kostrzewa, L. Liu, F. Pittler et al., *Hadron-Hadron Interactions from $N_f = 2+1+1$ Lattice QCD: $I = 3/2$ πK Scattering Length*, *Phys. Rev. D* **98** (2018) 114511 [1809.08886].

[97] [ETM 17G] C. Helmes, C. Jost, B. Knippschild, B. Kostrzewa, L. Liu, C. Urbach et al., *Hadron-Hadron Interactions from $N_f = 2+1+1$ lattice QCD: Isospin-1 KK scattering length*, *Phys. Rev. D* **96** (2017) 034510 [1703.04737].

[98] [PNDME 18] R. Gupta, Y.-C. Jang, B. Yoon, H.-W. Lin, V. Cirigliano and T. Bhattacharya, *Isovector Charges of the Nucleon from 2+1+1-flavor Lattice QCD*, *Phys. Rev.* **D98** (2018) 034503 [1806.09006].

[99] [CalLat 18] C. C. Chang et al., *A per-cent-level determination of the nucleon axial coupling from quantum chromodynamics*, *Nature* (2018) [1805.12130].

[100] [CalLat 19] A. Walker-Loud et al., *Lattice QCD Determination of g_A*, *PoS* **CD2018** (2020) 020 [1912.08321].

[101] [χQCD 18] J. Liang, Y.-B. Yang, T. Draper, M. Gong and K.-F. Liu, *Quark spins and Anomalous Ward Identity*, *Phys. Rev.* **D98** (2018) 074505 [1806.08366].

[102] [Mainz 19] T. Harris, G. von Hippel, P. Junnarkar, H.B. Meyer, K. Ottnad, J. Wilhelm et al., *Nucleon isovector charges and twist-2 matrix elements with $N_f = 2+1$ dynamical Wilson quarks*, *Phys. Rev. D* **100** (2019) 034513 [1905.01291].

[103] [PNDME 18A] H.-W. Lin, R. Gupta, B. Yoon, Y.-C. Jang and T. Bhattacharya, *Quark contribution to the proton spin from 2+1+1-flavor lattice QCD*, *Phys. Rev.* **D98** (2018) 094512 [1806.10604].

[104] [BMW 11A] S. Dürr et al., *Sigma term and strangeness content of octet baryons*, *Phys. Rev.* **D85** (2012) 014509 [1109.4265], [Erratum: Phys. Rev.D93,no.3,039905(2016)].

[105] [BMW 15] S. Dürr et al., *Lattice computation of the nucleon scalar quark contents at the physical point*, *Phys. Rev. Lett.* **116** (2016) 172001 [1510.08013].

[106] [χQCD 15A] Y.-B. Yang, A. Alexandru, T. Draper, J. Liang and K.-F. Liu, *πN and strangeness sigma terms at the physical point with chiral fermions*, *Phys. Rev.* **D94** (2016) 054503 [1511.09089].

[107] [QCDSF 12] G. Bali, P. Bruns, S. Collins, M. Deka, B. Glasle et al., *Nucleon mass and sigma term from lattice QCD with two light fermion flavors*, *Nucl.Phys.* **B866** (2013) 1 [1206.7034].

[108] [MILC 12C] W. Freeman and D. Toussaint, *Intrinsic strangeness and charm of the nucleon using improved staggered fermions*, *Phys. Rev.* **D88** (2013) 054503 [1204.3866].

[109] P. Junnarkar and A. Walker-Loud, *Scalar strange content of the nucleon from lattice QCD*, *Phys. Rev.* **D87** (2013) 114510 [1301.1114].

[110] [PNDME 18B] R. Gupta, B. Yoon, T. Bhattacharya, V. Cirigliano, Y.-C. Jang and H.-W. Lin, *Flavor diagonal tensor charges of the nucleon from (2+1+1)-flavor lattice QCD*, *Phys. Rev.* **D98** (2018) 091501 [1808.07597].

[111] [CalLat 20A] N. Miller et al., *Scale setting the Möbius domain wall fermion on gradient-flowed HISQ action using the omega baryon mass and the gradient-flow scales t_0 and w_0*, *Phys. Rev. D* **103** (2021) 054511 [2011.12166].

[112] [MILC 15] A. Bazavov et al., *Gradient flow and scale setting on MILC HISQ ensembles*, *Phys. Rev.* **D93** (2016) 094510 [1503.02769].

[113] [CLS 16] M. Bruno, T. Korzec and S. Schaefer, *Setting the scale for the CLS 2+1 flavor ensembles*, *Phys. Rev.* **D95** (2017) 074504 [1608.08900].

[114] [BMW 12A] S. Borsanyi, S. Dürr, Z. Fodor, C. Hoelbling, S.D. Katz et al., *High-precision scale setting in lattice QCD*, *JHEP* **1209** (2012) 010 [1203.4469].

[115] [BMW 20] Sz. Borsanyi et al., *Leading hadronic contribution to the muon magnetic moment from lattice QCD*, *Nature* **593** (2021) 51 [2002.12347].

[116] [HotQCD 14] A. Bazavov et al., *Equation of state in (2+1)-flavor QCD*, *Phys.Rev.* **D90** (2014) 094503 [1407.6387].

[117] [RBC/UKQCD 10A] Y. Aoki et al., *Continuum limit physics from 2+1 flavor domain wall QCD*, *Phys.Rev.* **D83** (2011) 074508 [1011.0892].

[118] [HPQCD 05B] A. Gray et al., *The upsilon spectrum and m_b from full lattice QCD*, *Phys.Rev.* **D72** (2005) 094507 [hep-lat/0507013].

[119] C. Aubin et al., *Light hadrons with improved staggered quarks: Approaching the continuum limit*, *Phys. Rev.* **D70** (2004) 094505 [hep-lat/0402030].

[120] [HPQCD 09B] C. T. H. Davies, E. Follana, I. Kendall, G.P. Lepage and C. McNeile, *Precise determination of the lattice spacing in full lattice QCD*, *Phys.Rev.* **D81** (2010) 034506 [0910.1229].

[121] K. Symanzik, *Continuum limit and improved action in lattice theories. 1. Principles and ϕ^4 theory*, Nucl. Phys. **B226** (1983) 187.

[122] K. Symanzik, *Continuum limit and improved action in lattice theories. 2. O(N) nonlinear sigma model in perturbation theory*, Nucl. Phys. **B226** (1983) 205.

[123] [RBC 07A] D. J. Antonio et al., *Localization and chiral symmetry in 3 flavor domain wall QCD*, Phys. Rev. **D77** (2008) 014509 [0705.2340].

[124] [MILC 10] A. Bazavov et al., *Topological susceptibility with the asqtad action*, Phys. Rev. **D81** (2010) 114501 [1003.5695].

[125] [ALPHA 10C] S. Schaefer, R. Sommer and F. Virotta, *Critical slowing down and error analysis in lattice QCD simulations*, Nucl.Phys. **B845** (2011) 93 [1009.5228].

[126] M. Lüscher, *Topology, the Wilson flow and the HMC algorithm*, *PoS* **LATTICE2010** (2010) 015 [1009.5877].

[127] S. Schaefer, *Algorithms for lattice QCD: progress and challenges*, AIP Conf.Proc. **1343** (2011) 93 [1011.5641].

[128] A. Chowdhury, A. Harindranath, J. Maiti and P. Majumdar, *Topological susceptibility in lattice Yang-Mills theory with open boundary condition*, JHEP **02** (2014) 045 [1311.6599].

[129] [LSD 14] R. C. Brower et al., *Maximum-Likelihood Approach to Topological Charge Fluctuations in Lattice Gauge Theory*, Phys. Rev. **D90** (2014) 014503 [1403.2761].

[130] [JLQCD 15] H. Fukaya, S. Aoki, G. Cossu, S. Hashimoto, T. Kaneko and J. Noaki, *η' meson mass from topological charge density correlator in QCD*, Phys. Rev. **D92** (2015) 111501 [1509.00944].

[131] L. Del Debbio, H. Panagopoulos and E. Vicari, *Theta dependence of SU(N) gauge theories*, JHEP **08** (2002) 044 [hep-th/0204125].

[132] C. Bernard et al., *Topological susceptibility with the improved Asqtad action*, Phys. Rev. **D68** (2003) 114501 [hep-lat/0308019].

[133] M. Lüscher and S. Schaefer, *Lattice QCD without topology barriers*, JHEP **1107** (2011) 036 [1105.4749].

[134] M.G. Endres, R.C. Brower, W. Detmold, K. Orginos and A.V. Pochinsky, *Multiscale Monte Carlo equilibration: Pure Yang-Mills theory*, Phys. Rev. **D92** (2015) 114516 [1510.04675].

[135] W. Detmold and M.G. Endres, *Scaling properties of multiscale equilibration*, Phys. Rev. **D97** (2018) 074507 [1801.06132].

[136] S. Mages, B.C. Toth, S. Borsanyi, Z. Fodor, S. Katz and K.K. Szabo, *Lattice QCD on Non-Orientable Manifolds*, Phys. Rev. **D95** (2017) 094512 [1512.06804].

[137] M. Bruno et al., *Simulation of QCD with $N_f = 2 + 1$ flavors of non-perturbatively improved Wilson fermions*, JHEP **02** (2015) 043 [1411.3982].

[138] R. Brower, S. Chandrasekharan, J.W. Negele and U. Wiese, *QCD at fixed topology*, *Phys.Lett.* **B560** (2003) 64 [hep-lat/0302005].

[139] S. Aoki, H. Fukaya, S. Hashimoto and T. Onogi, *Finite volume QCD at fixed topological charge*, *Phys. Rev.* **D76** (2007) 054508 [0707.0396].

[140] I. Bautista, W. Bietenholz, A. Dromard, U. Gerber, L. Gonglach, C.P. Hofmann et al., *Measuring the Topological Susceptibility in a Fixed Sector*, *Phys. Rev.* **D92** (2015) 114510 [1503.06853].

[141] W. Bietenholz, C. Czaban, A. Dromard, U. Gerber, C.P. Hofmann, H. Mejía-Díaz et al., *Interpreting Numerical Measurements in Fixed Topological Sectors*, *Phys. Rev.* **D93** (2016) 114516 [1603.05630].

[142] C. Bernard and D. Toussaint, *Effects of nonequilibrated topological charge distributions on pseudoscalar meson masses and decay constants*, *Phys. Rev.* **D97** (2018) 074502 [1707.05430].

[143] S. Duane, A.D. Kennedy, B.J. Pendleton and D. Roweth, *Hybrid Monte Carlo*, *Phys. Lett.* **B195** (1987) 216.

[144] M.A. Clark and A.D. Kennedy, *Accelerating staggered fermion dynamics with the rational hybrid Monte Carlo (RHMC) algorithm*, *Phys. Rev.* **D75** (2007) 011502 [hep-lat/0610047].

[145] [MILC 12B] A. Bazavov et al., *Lattice QCD ensembles with four flavors of highly improved staggered quarks*, *Phys.Rev.* **D87** (2013) 054505 [1212.4768].