3 General definition of the low-energy limit of the Standard Model

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This section discusses the matching of quantum chromodynamics (QCD) and quantum electrodynamics (QED) to nature in order to obtain predictions for low-energy Standard Model observables. In particular, we discuss the prescription dependence, i.e., the dependence on which observables are matched, arising when one neglects electromagnetic interactions, an approximation made in numerous lattice and phenomenological calculations. These ambiguities need to be controlled when combining high-precision observables—typically with less than 1% of relative uncertainty—in that approximation. In order to facilitate that, we propose here a fixed prescription for the separation of QCD and QED contributions to any given hadronic observable. While this prescription is, in principle, arbitrary, one has to take care not to introduce artificially large QED contributions and to stay close to prescriptions used commonly in phenomenology. This prescription was discussed and agreed upon during an open workshop that took place at the Higgs Centre for Theoretical Physics, Edinburgh, in May 2023, and therefore is referred to as the "Edinburgh Consensus."[1]

We note that since this consensus emerged only recently, the majority of results in this review are averaged neglecting potential discrepancies arising from the ambiguities. This is, on the one hand, an adequate procedure in the case of quantities with uncertainties larger than the size of expected QED corrections. On the other hand, it can be difficult to correct these ambiguities to a common prescription since it requires the knowledge of derivatives of observables in quark masses and couplings, rarely communicated in papers. We emphasize the present consensus in the hope that it will be widely adopted in upcoming high-precision Standard Model predictions, allowing future editions of this review to avoid uncertainties resulting from these ambiguities.

3.1 First-order isospin-breaking expansion

According to our present knowledge, hadronic physics is well described by the low-energy limit of the Standard Model, which is understood as energies well below the electroweak symmetry-breaking scale $\mathcal{S}_{\mathrm{ESB}} \approx 100$ GeV. In that limit, the Standard Model is an $\mathrm{SU}(3) \times \mathrm{U}(1)$ gauge theory defined by the QCD+QED Lagrangian, whose free parameters are the e-, μ-, and τ-lepton masses, the u-, d-, s-, c-, and b-quark masses, and the strong and electromagnetic coupling constants, respectively, g_s and e. In that context, isospin symmetry is defined by assuming that the up and down quarks are identical particles apart from their flavour. This symmetry is only approximate and it is broken by two effects: the small but different masses of the two quarks, and their different electric charges. The total effect is expected to be small, typically a $\mathcal{O}(1\%)$ perturbation of a hadronic energy or amplitude. Therefore, we consider only first-order perturbations in isospin-breaking effects, and we expect this approximation to be accurate at the level of $\mathcal{O}(10^{-4})$ relative precision.

The asymptotic states of QCD are hadrons not quarks, and hadron properties are the only unambiguous observables experimentally available. Similarly, the strong coupling constant is not directly accessible and can be substituted through dimensional transmutation by a

[1] https://indico.ph.ed.ac.uk/event/257/

dimensionful hadronic energy scale. Moreover, the running of the electromagnetic coupling constant is a higher-order correction beyond the order considered here. It can be fixed to its Thomson-limit value. Finally, nature can be reproduced (up to weak and gravitational effects) by fixing the bare parameters of the QCD+QED Lagrangian to reproduce the following inputs:

1. the Thomson-limit constant $\alpha^\phi = \frac{e^2}{4\pi} = 7.2973525693(11) \times 10^{-3}$ [1],
2. the experimentally observed lepton masses m_ℓ^ϕ,
3. a choice of N_f known independent hadronic quantities M^ϕ, setting the quark masses,
4. a single known dimensionful hadronic quantity $\mathcal{S}^\phi$, setting the QCD scale.

The vectors m_ℓ and M have three and N_f components, respectively, where N_f is the number of quark flavours in the calculation. In the present context, "known" is understood as experimentally known for measurable quantities, or theoretically predicted for more abstract quantitities, which are not accessible experimentally, but are renormalized and gauge invariant and can be predicted by lattice gauge theory. If the dependency of a given observable $X(\alpha, m_\ell, M, \mathcal{S})$ on the above variables is known, then its physical value is predicted by

$$X^\phi = (\mathcal{S}^\phi)^{[X]}\tilde{X}(\alpha^\phi, m_\ell^\phi/\mathcal{S}^\phi, M^\phi/\mathcal{S}^\phi) \equiv X(\alpha^\phi, m_\ell^\phi, M^\phi, \mathcal{S}^\phi)\,, \tag{22}$$

where $\tilde{X}$ is the dimensionless function describing X in units of the scale $\mathcal{S}$, and $[X]$ is the energy dimension of X. Here M and $\mathcal{S}$ are assumed, without loss of generality, to have an energy dimension of 1. Due to the renormalizability of QCD+QED, this prediction is unambiguous, i.e., changing the variables M^ϕ and $\mathcal{S}^\phi$ to other inputs with known physical values will lead to the same prediction for renormalized observables.[2]

In many instances, the precision required on hadronic observables is not as small as one percent, and isospin-breaking effects are potentially negligible. In those cases, it is generally considerably simpler to neglect the QED contributions, both for lattice and phenomenological calculations. Moreover, even for observables requiring isospin-breaking corrections to be computed, it can be phenomenologically relevant to separate an isospin-symmetric value and isospin-breaking corrections (e.g., specific parts of the HVP contribution to the muon $g-2$, decay constants in weak decays). However, since experimental measurements always contain isospin-breaking corrections, there are no experimental result available to define the list of inputs above for $\alpha = 0$, or in the isospin-symmetric limit. Still, one would like to define an expansion of the form

$$X^\phi = \bar{X} + X_\gamma + X_{\mathrm{SU}(2)}\,, \tag{23}$$

where $\bar{X}$ is the isospin-symmetric value of X, and X_γ and $X_{\mathrm{SU}(2)}$ are the first-order electromagnetic and strong isospin-breaking corrections, respectively. Only the sum of these three terms is unambiguous.[3] Defining a value for individual terms is prescription-dependent, and requires additional, in principle arbitrary, inputs. This issue has been discussed in reviews [2, 3], and both the phenomenology [4–6] and lattice [7–20] literature. If quantities defined at $\alpha = 0$ are involved in the investigation of anomalies related to new physics searches, *the associated prescriptions must be matched across predictions.* In the next section, we propose a prescription agreed upon at the dedicated May 2023 workshop in Edinburgh.

[2]Here "renormalizability" for QED is understood as perturbative renormalizability, which is sufficient in this context.

[3]Here "unambiguous" is used in a loose sense. Ambiguities of the order $\mathcal{O}(1/m_Z)$ and $\mathcal{O}(1/m_{N_f+1})$, as well as higher-order isospin-breaking corrections, remain and are considered to be irrelevant.

	QCD	isoQCD
M_{π^+}	135.0 MeV	135.0 MeV
M_{K^+}	491.6 MeV	494.6 MeV
M_{K^0}	497.6 MeV	494.6 MeV
$M_{D_s^+}$	1967 MeV	1967 MeV
$M_{B_s^0}$	5367 MeV	5367 MeV
f_{π^+}	130.5 MeV	130.5 MeV

	QCD	isoQCD
M_{π^+}/f_{π^+}	1.034	1.034
M_{K^+}/f_{π^+}	3.767	3.790
M_{K^0}/f_{π^+}	3.813	3.790
$M_{D_s^+}/f_{\pi^+}$	15.07	15.07
$M_{B_s^0}/f_{\pi^+}$	41.13	41.13

Table 5: Edinburgh Consensus for the definition of pure QCD and isospin-symmetric QCD. The rightmost table is redundant and provided for convenience.

3.2 Edinburgh Consensus

The decomposition Eq. (23) can be unambiguously defined given two extra sets of inputs $(\hat{m}_\ell, \hat{M}, \hat{\mathcal{S}})$ and $(\bar{m}_\ell, \bar{M}, \bar{\mathcal{S}})$ specifying pure QCD and isospin-symmetric QCD, respectively (denoted QCD and isoQCD). It is understood that in QCD isospin symmetry can still be broken by the up-down quark-mass difference. The QCD and isoQCD values of an observable X can then be defined by

$$\hat{X} = X(0, \hat{m}_\ell, \hat{M}, \hat{\mathcal{S}}) \qquad \text{and} \qquad \bar{X} = X(0, \bar{m}_\ell, \bar{M}, \bar{\mathcal{S}}) \,, \tag{24}$$

respectively. The variables $\bar{M}$, $\bar{\mathcal{S}}$ must have one dimension of linear dependency to reflect the exact isospin symmetry of this theory. This means that there are only N_f independent numbers. Finally, the corrections in Eq. (23) are then defined by

$$X_\gamma = X^\phi - \hat{X} \qquad \text{and} \qquad X_{\mathrm{SU}(2)} = \hat{X} - \bar{X} \,. \tag{25}$$

One should notice that these definitions already constitute in themselves a prescription, as QED has an isospin-symmetric component which is here assumed to be excluded from the component $\bar{X}$.

The proposed prescription defines lepton masses to always be equal to their experimental values (for which negligible experimental uncertainties are discarded), i.e., $\hat{m}_\ell = \bar{m}_\ell = m_\ell^\phi$, and is based on the mass variables $M = (M_{\pi^+}, M_{K^+}, M_{K^0}, M_{D_s^+}, M_{B_s^0})$ and the scale-setting quantity f_{π^+}, with the values given in Tab. 5.[4] We will now comment on the definition and applications of that prescription.

3.3 Comparison to other schemes

The hadronic quantities that define the proposed prescription, as well as their input values, have been chosen to balance between two main constraints, on the one hand numerical and on the other hand theoretical. Since any uncertainties on the theoretical inputs have to be propagated to the predictions, the numerical constraint requires choosing the matching observables among those that can be computed on the lattice with the highest accuracy. The theoretical constraint requires choosing a definition of QCD that leads to isospin-breaking corrections which are as close as possible to what has commonly been done in the past, in particular, in phenomenological calculations.

[4] For calculations with no active c and/or b quarks, the $M_{D_s^+}$ and/or $M_{B_s^0}$ components should be ignored.

On the numerical side, all the chosen hadronic inputs can be extracted from the leading exponential behaviour at large Euclidean times of two-point mesonic lattice correlators with high numerical precision. This constraint is the main reason behind the choice of f_{π^+} as the scale-setting observable. From the theoretical and phenomenological perspectives, this can be seen as an uncomfortable choice. Indeed, the physical quantity that is measured in experiments is the leptonic decay rate of the charged pion. In the full theory (QCD+QED) soft photons as well as nonfactorisable virtual QED corrections have to be taken into account in the theoretical calculation in order to use the experimental values as an input, and previous knowledge of the CKM matrix element V_{ud} is required. From this perspective, for example, the choice of the Ω^--baryon mass used by several lattice collaborations might be more natural. However, the majority of lattice calculations are still performed in the $\alpha = 0$ limit, which makes f_{π^+} a more accessible choice than a baryonic quantity in most cases. It is crucial to note that our prescription defines QCD and isoQCD in the space of possible $\alpha = 0$ theories, but the choice of coordinates to define these points is arbitrary and can be changed using standard change-of-variable algebra, while keeping the prescription fixed. In particular, the scale setting variable can be changed, as we discuss now.

The prescription above can be implemented by using other inputs. This is possible because QCD is renormalizable. Indeed, one can start by defining QCD using our prescription to compute $\hat{X}$ and $\hat{M}_\Omega$, following the notation of the previous section, namely

$$\hat{X} = X(0, \hat{m}_\ell, \hat{M}, \hat{f}_{\pi^+}) \qquad \text{and} \qquad \hat{M}_\Omega = M_\Omega(0, \hat{m}_\ell, \hat{M}, \hat{f}_{\pi^+})\,, \tag{26}$$

where $\hat{M}$ and $\hat{f}_{\pi^+}$ are given by the "QCD" column in Tab. 5. Once this calculation has been done, the value of $\hat{M}_\Omega$ that has been obtained (assuming for the moment that the errors are negligible) can be substituted to $\hat{f}_{\pi^+}$ to redefine our prescription independently from the pion decay constant. In practice, though, it will not be possible to neglect the errors on $\hat{M}_\Omega$. This means that the equivalence between the two sets of coordinates, explicitly

$$\hat{X} = X(0, \hat{m}_\ell, \hat{M}, \hat{f}_{\pi^+}) = X(0, \hat{m}_\ell, \hat{M}, \hat{M}_\Omega)\,, \tag{27}$$

can be established within the errors on $\hat{M}_\Omega$ that will have to be propagated on any prediction. In this respect, the choice of defining QCD by prescribing with no errors the values appearing in Tab. 5 puts the choice of $\hat{f}_{\pi^+}$ on a slightly different footing than $\hat{M}_\Omega$. The accuracy of this matching will directly depend on the accuracy of the dimensionless ratio $\hat{M}_\Omega/\hat{f}_{\pi^+}$. The whole discussion above can be reiterated identically for isoQCD, replacing hatted quantities ($\hat{X}$, ...) with barred ones ($\bar{X}$, ...). It is important to note that f_{π^+} is used only to define QCD and plays no role in defining the full QCD+QED theory. In particular, through a change of scale variable, like that discussed above, one does not need to know the QED correction to the π^+ leptonic decay rate to use our prescription, and one does not lose the ability to predict this rate for high-precision determinations of the $|V_{ud}|$ CKM matrix element.

Theoretical constraints are the main reason behind the particular choice of values prescribed in Tab. 5. Most isospin-breaking separation schemes used in the literature aim at keeping constant the value of a definition of the renormalized quark masses when sending α to zero between the physical QCD+QED theory and QCD. Such a class of constraints was implemented in various ways, for example by the RM123/RM123S collaboration by computing directly quark masses in the $\overline{\text{MS}}$ scheme at 2 GeV [8, 15, 21, 22]. Another example comes from the BMW collaboration, which used in several calculations [7, 12, 16, 20] a scheme defined by keeping fixed the squared masses of $\bar{q}q$-connected mesons when changing α. Although these schemes share similar aims, they are not equivalent and differ by the choice of renormalization scale and scheme, as well as the contribution from higher-order chiral corrections when using squared meson masses. However, at the level of precision of current lattice calculations, no significant discrepancies were observed between both approaches [15, 19, 20, 23], and the numerical values of the pion and kaon masses in Tab. 5 are compatible with these determinations within the current level of precision. We also note that the mass values prescribed here are compatible with those produced from phenomenological inputs in the first edition of FLAG [24], which predates the lattice references quoted above.

We end this chapter with a comment on Gasser-Rusetsky-Scimemi (GRS) type schemes [5]. These authors emphasized the importance of keeping track of the scheme dependence of the splitting in Eq. (23). They furthermore proposed to keep renormalized quark masses and the strong coupling at a particular matching scale μ_1 (and a chosen renormalization scheme) fixed as one turns off the electromagnetic coupling. In contrast to the perturbative models studied by GRS, such a scheme is hard to implement in QCD. Even on the lattice, uncertainties are introduced which are larger than the isospin-breaking corrections (see the sections on quark masses and α_s). The RM123S scheme [21] mentioned above is an electro-quenched GRS type scheme.[5] Since there are no electromagnetic contributions to α_s in the electro-quenched approximation, the generic difficulties of a GRS type scheme are circumvented.

References

[1] Particle Data Group collaboration, *Review of Particle Physics*, *PTEP* **2022** (2022) 083C01.

[2] T. Aoyama et al., *The anomalous magnetic moment of the muon in the Standard Model*, *Phys. Rept.* **887** (2020) 1 [2006.04822].

[3] N. Tantalo, *Matching lattice QC+ED to Nature*, *PoS* **LATTICE2022** (2023) 249 [2301.02097].

[4] J. Gasser and H. Leutwyler, *Quark masses*, *Phys. Rept.* **87** (1982) 77.

[5] J. Gasser, A. Rusetsky and I. Scimemi, *Electromagnetic corrections in hadronic processes*, *Eur. Phys. J.* **C32** (2003) 97 [hep-ph/0305260].

[6] J. Gasser and G.R.S. Zarnauskas, *On the pion decay constant*, *Phys. Lett.* **B693** (2010) 122 [1008.3479].

[7] [BMW 13A] Sz. Borsanyi et al., *Isospin splittings in the light baryon octet from lattice QCD and QED*, *Phys. Rev. Lett.* **111** (2013) 252001 [1306.2287].

[8] [RM123 13] G. M. de Divitiis, R. Frezzotti, V. Lubicz, G. Martinelli, R. Petronzio et al., *Leading isospin breaking effects on the lattice*, *Phys.Rev.* **D87** (2013) 114505 [1303.4896].

[9] N. Tantalo, *Isospin Breaking Effects on the Lattice*, *PoS* **LATTICE2013** (2014) 007 [1311.2797].

[10] A. Portelli, *Inclusion of isospin breaking effects in lattice simulations*, *PoS* **LATTICE2014** (2015) 013 [1505.07057].

[5] The electro-quenched approximation is defined by setting the electric charges of the sea quarks to zero.

[11] [QCDSF/UKQCD 15A] R. Horsley et al., *QED effects in the pseudoscalar meson sector*, *JHEP* **04** (2016) 093 [1509.00799].

[12] [BMW 16A] Z. Fodor, C. Hoelbling, S. Krieg, L. Lellouch, T. Lippert, A. Portelli et al., *Up and down quark masses and corrections to Dashen's theorem from lattice QCD and quenched QED*, *Phys. Rev. Lett.* **117** (2016) 082001 [1604.07112].

[13] A. Bussone et al., *On the definition of schemes for computing leading order isospin breaking corrections*, *PoS* **LATTICE2018** (2018) 293 [1810.11647].

[14] [MILC 18] S. Basak et al., *Lattice computation of the electromagnetic contributions to kaon and pion masses*, *Phys. Rev.* **D99** (2019) 034503 [1807.05556].

[15] M. Di Carlo, D. Giusti, V. Lubicz, G. Martinelli, C. Sachrajda, F. Sanfilippo et al., *Light-meson leptonic decay rates in lattice QCD+QED*, *Phys. Rev. D* **100** (2019) 034514 [1904.08731].

[16] [BMW 20] Sz. Borsanyi et al., *Leading hadronic contribution to the muon magnetic moment from lattice QCD*, *Nature* **593** (2021) 51 [2002.12347].

[17] C.T. Sachrajda, *Isospin Breaking in Lattice QCD Computations of Decay Amplitudes*, *Acta Phys. Polon. B* **52** (2021) 175 [2104.04312].

[18] M. Cè et al., *Window observable for the hadronic vacuum polarization contribution to the muon g-2 from lattice QCD*, *Phys. Rev. D* **106** (2022) 114502 [2206.06582].

[19] P. Boyle et al., *Isospin-breaking corrections to light-meson leptonic decays from lattice simulations at physical quark masses*, *JHEP* **02** (2023) 242 [2211.12865].

[20] A. Boccaletti et al., *High precision calculation of the hadronic vacuum polarisation contribution to the muon anomaly*, 2407.10913.

[21] [RM123 11] G. M. de Divitiis, P. Dimopoulos, R. Frezzotti, V. Lubicz, G. Martinelli et al., *Isospin breaking effects due to the up-down mass difference in lattice QCD*, *JHEP* **1204** (2012) 124 [1110.6294].

[22] D. Giusti, V. Lubicz, G. Martinelli, C.T. Sachrajda, F. Sanfilippo, S. Simula et al., *First lattice calculation of the QED corrections to leptonic decay rates*, *Phys. Rev. Lett.* **120** (2018) 072001 [1711.06537].

[23] A. Portelli, *Estimation of scheme ambiguities in the separation of isospin-breaking effects in lattice QCD calculations*, *Lattice 2022*, 2022, https://indico.hiskp.uni-bonn.de/event/40/contributions/542/.

[24] [FLAG 10] G. Colangelo, S. Dürr, A. Jüttner, L. Lellouch, H. Leutwyler et al., *Review of lattice results concerning low energy particle physics*, *Eur.Phys.J.* **C71** (2011) 1695 [1011.4408].