# C Notes

In the following Appendices we provide more detailed information on the simulations used to calculate the quantities discussed in Secs. 4–11. We present this information only for results that are new w.r.t. FLAG 21. For all other results the information is available in the corresponding Appendices C.1–C.9 in FLAG 21 [1], B.1–B.8 in FLAG 19 [2], and B.1–B.7 in FLAG 16 [3].

## C.1 Notes to Sec. 4 on quark masses

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| CLQCD 23 | [4] | 2+1 | 0.052, 0.077, 0.11 | smeared Wilson-clover/Symanzik |

Table 80: Continuum extrapolations/estimation of lattice artifacts in determinations of $m_{ud}$, $m_s$ and, in some cases $m_u$ and $m_d$, with $N_f = 2+1$ quark flavours.

| Collab. | Ref. | $N_f$ | $M_{\pi,\mathrm{min}}$ [MeV] | Description |
|---|---|---|---|---|
| CLQCD 23 | [4] | 2+1 | 135.5 | |

Table 81: Chiral extrapolation/minimum pion mass in determinations of $m_{ud}$, $m_s$, and in some cases $m_u$ and $m_d$, with $N_f = 2+1$ quark flavours.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\mathrm{min}}L$ | Description |
|---|---|---|---|---|---|
| CLQCD 23 | [4] | 2+1 | 2.5 – 5.1 | 3.45 | |

Table 82: Finite-volume effects in determinations of $m_{ud}$, $m_s$ and, in some cases $m_u$ and $m_d$, with $N_f = 2+1$ quark flavours.

| Collab. | Ref. | $N_f$ | Description |
|---|---|---|---|
| CLQCD 23 | [4] | 2+1 | RI/MOM |

Table 83: Renormalization in determinations of $m_{ud}$, $m_s$ and, in some cases $m_u$ and $m_d$, with $N_f = 2+1$ quark flavours.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| ALPHA 23 | [5] | 2+1 | 0.085, 0.075, 0.064, 0.049 | $\mathcal{O}(a^2)$ terms, with mass-dependent coefficients, are included in the chiral-continuum extrapolation. $t_0$ is used as intermediate scale with the physical scale set by a combination of $f_\pi$ and $f_K$ in the isosymmetric limit. |

Table 84: Continuum extrapolations/estimation of lattice artifacts in the determinations of $m_c$ with $N_f = 2+1$ quark flavours.

| Collab. | Ref. | $N_f$ | $M_{\pi,\text{min}}$ [MeV] | Description |
|---|---|---|---|---|
| ALPHA 23 | [5] | 2+1 | 200 | |

Table 85: Chiral extrapolation/minimum pion mass in the determinations of $m_c$ with $N_f = 2+1$ quark flavours.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\text{min}}L$ | Description |
|---|---|---|---|---|---|
| ALPHA 23 | [5] | 2+1 | 2.7, 2.4, 3.1/4.1, 2.4/3.1 | 3.9, 5.1, 4.2, 4.1 | No explicit discussion of FSE. |

Table 86: Finite-volume effects in the determinations of $m_c$ with $N_f = 2+1$ quark flavours.

| Collab. | Ref. | $N_f$ | Description |
|---|---|---|---|
| ALPHA 23 | [5] | 2+1 | Schrödinger functional |

Table 87: Renormalization in the determinations of $m_c$ with $N_f = 2 + 1$ quark flavours.

## C.2 Notes to Sec. 5 on $|V_{ud}|$ and $|V_{us}|$

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| PACS 22 | [6] | 2+1 | 0.085, 0.063 | Nonperturbative $\mathcal{O}(a)$ clover quark action. Scale set from Ξ-baryon mass. |

Table 88: Continuum extrapolations/estimation of lattice artifacts in the determinations of $f_+(0)$.

| Collab. | Ref. | $N_f$ | $M_{\pi,\rm min}$ [MeV] | Description |
|---|---|---|---|---|
| PACS 22 | [6] | 2+1 | 135 | Physical point simulation at a single pion mass 135 MeV. |

Table 89: Chiral extrapolation/minimum pion mass in determinations of $f_+(0)$.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\rm min}L$ | Description |
|---|---|---|---|---|---|
| PACS 22 | [6] | 2+1 | 10.9 | 7.5 | |

Table 90: Finite-volume effects in determinations of $f_+(0)$.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| ETM 21 | [7] | 2+1+1 | 0.07, 0.08, 0.09 | Wilson-clover twisted mass quark action. Relative scale through gradient flow scale $w_0$ and absolute scale through $f_\pi$. |

Table 91: Continuum extrapolations/estimation of lattice artifacts in determinations of $f_K/f_\pi$.

| Collab. | Ref. | $N_f$ | $M_{\pi,\rm min}$ [MeV] | Description |
|---|---|---|---|---|
| ETM 21 | [7] | 2+1+1 | 134 | Chiral extrapolation based on NLO SU(2) $\chi$PT. |

Table 92: Chiral extrapolation/minimum pion mass in determinations of $f_K/f_\pi$.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\mathrm{min}}L$ | Description |
|---|---|---|---|---|---|
| ETM 21 | [7] | 2+1+1 | 2.0–5.6 | 3.8 | Three different volumes at $M_\pi =$ 253 MeV and $a = 0.08$ fm. |

Table 93: Finite-volume effects in determinations of $f_K/f_\pi$.

## C.3 Notes to section 6 on Kaon mixing

### C.3.1 Kaon $B$-parameter $B_K$

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| RBC/UKQCD 24 | [8] | 2+1 | 0.114, 0.084, 0.073 | Combined continuum and chiral (NLO. SU(2)) extrapolation fits. Assigned systematic error at the per-mille level |

Table 94: Continuum extrapolations/estimation of lattice artifacts in determinations of $B_K$.

| Collab. | Ref. | $N_f$ | $M_{\pi,\min}$ [MeV] | Description |
|---|---|---|---|---|
| RBC/UKQCD 24 | [8] | 2+1 | 139, 139, 232 | Chiral extrapolations based on SU(2)-$\chi$PT fits at NLO. Systematic uncertainties amount to less than half a per cent. |

Table 95: Chiral extrapolation/minimum pion mass in determinations of $B_K$.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\min}L$ | Description |
|---|---|---|---|---|---|
| RBC/UKQCD 24 | [8] | 2+1 | 5.5, 5.4, 3.5, 2.6 | 3.9, 3.8, 4.1 | Finite-volume effects are found to be negligible compared to other systematic effects and are thus omitted in the final error budget. |

Table 96: Finite-volume effects in determinations of $B_K$.

| Collab. | Ref. | $N_f$ | Ren. | running match. | Description |
|---|---|---|---|---|---|
| RBC/UKQCD 24 | [8] | 2+1 | RI | PT1$\ell$ | Two different RI-SMOM schemes used to estimate a 1% systematic error owing to the perturbative matching to $\overline{\text{MS}}$. |

Table 97: Running and matching in determinations of $B_K$.

### C.3.2 Kaon BSM $B$-parameters

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| RBC/UKQCD 24 | [8] | 2+1 | 0.114, 0.084, 0.073 | Systematic uncertainties ranging from a minimum of 0.4% (for the case of $B_2$) to 1.9% (for the case of $B_3$). |

Table 98: Continuum extrapolations/estimation of lattice artifacts in determinations of the BSM $B_i$ parameters.

| Collab. | Ref. | $N_f$ | $M_{\pi,\rm min}$ [MeV] | Description |
|---|---|---|---|---|
| RBC/UKQCD 24 | [8] | 2+1 | 139, 139, 232 | Chiral extrapolations based on SU(2)-$\chi$PT fits at NLO. Systematic uncertainties amount to less than half a percent. |

Table 99: Chiral extrapolation/minimum pion mass in determinations of the BSM $B_i$ parameters.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\rm min}L$ | Description |
|---|---|---|---|---|---|
| RBC/UKQCD 24 | [8] | 2+1 | 5.5, 5.4, 3.5 2.6 | 3.9, 3.8, 4.1 | Finite-volume effects are at most at the 2 per-mille level. They are negligible compared to other systematic effects and are therefore omitted in the error budget. |

Table 100: Finite-volume effects in determinations of the BSM $B_i$ parameters.

| Collab. | Ref. | $N_f$ | Ren. | running match. | Description |
|---|---|---|---|---|---|
| RBC/UKQCD 24 | [8] | 2+1 | RI | PT1$\ell$ | Two different RI-SMOM schemes used to estimate the systematic error owing to the perturbative matching to $\overline{\rm MS}$; minimal value of about 0.7% for the case of $B_2$ and maximal of 2.4% for $B_3$. |

Table 101: Running and matching in determinations of the BSM $B_i$ parameters.

### C.3.3 $K \to \pi\pi$ decay amplitudes

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| RBC/UKQCD 23A | [9] | 2+1 | 0.193 | Single lattice spacing. |

Table 102: Continuum extrapolations/estimation of lattice artifacts in determinations of the $K \to \pi\pi$ decay amplitudes.

| Collab. | Ref. | $N_f$ | $M_{\pi,\text{min}}$ [MeV] | Description |
|---|---|---|---|---|
| RBC/UKQCD 23A | [9] | 2+1 | 142.6 | Single pion mass value, close to the physical point. |

Table 103: Chiral extrapolation/minimum pion mass in determinations of the $K \to \pi\pi$ decay amplitudes.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\text{min}}L$ | Description |
|---|---|---|---|---|---|
| RBC/UKQCD 23A | [9] | 2+1 | 4.6 | 3.3 | Finite-volume effects amount to a 7% systematic error contribution to the final error budget of $A_0$ and $A_2$. |

Table 104: Finite-volume effects in determinations of the $K \to \pi\pi$ decay amplitudes.

| Collab. | Ref. | $N_f$ | Ren. | running match. | Description |
|---|---|---|---|---|---|
| RBC/UKQCD 23A | [9] | 2+1 | RI | PT1$\ell$ | Two different RI-SMOM schemes are used. One of the two schemes is used for the final analysis. A systematic error ranging from 6% to 16%, depending on the considered case, is included based on the disperision of other sets of intermediate scheme and scales. Systematic uncertainties arising from the computation of the Wilson coefficients in the $\overline{\text{MS}}$ scheme amount to 12%. |

Table 105: Running and matching in determinations of the $K \to \pi\pi$ decay amplitudes.

### C.4 Notes to Sec. 7 on $D$-meson decay constants and form factors

| Collab. | Ref. | $N_f$ | $M_{\pi,\rm min}$ [MeV] | Description |
|---|---|---|---|---|
| ETM 13F<br>ETM 14E<br>ETM 21B | [10–12] | 2+1+1 | 245, 239, 211<br><br>167, 137, 134 | $f_{D_s}\sqrt{m_{D_s}}$ in ETM 13F and $f_{D_s}/m_{D_s}$ in ETM 14E are extrapolated using both a quadratic and a linear fit in $m_l$ plus $O(a^2)$ terms. In ETM 21B either $w_0$ or the $D_{(s)}$ meson mass are used as scaling variables in the chiral-continuum extrapolations. |

Table 106: Chiral extrapolation/minimum pion mass in $N_f = 2+1+1$ determinations of the $D$- and $D_s$-meson decay constants. For actions with multiple species of pions, masses quoted are the RMS pion masses. The different $M_{\pi,\rm min}$ entries correspond to the different lattice spacings.

| Collab. | Ref. | $N_f$ | $M_{\pi,\rm min}$ [MeV] | Description |
|---|---|---|---|---|
| RQCD/ALPHA 24 | [13] | 2+1 | 335, 129, 155, 130, 175, 337 | The dependence on light and strange quark masses is described using fit ansätze inspired by NLO HM$\chi$PT. |
| ALPHA 23 | [5] | 2+1 | 277, 415, 200, 257 | HM$\chi$PT expressions are used for the quantities $(8t_0)^{3/4} f_{D_{(s)}}\sqrt{m_{D_{(s)}}}$. |

Table 107: Chiral extrapolation/minimum pion mass in $N_f = 2+1$ determinations of the $D$- and $D_s$-meson decay constants. For actions with multiple species of pions, masses quoted are the RMS pion masses. The different $M_{\pi,\rm min}$ entries correspond to the different lattice spacings.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\rm min}L$ | Description |
|---|---|---|---|---|---|
| ETM 13F<br>ETM 14E<br>ETM 21B | [10–12] | 2+1+1 | 2.3/4.6,<br>2.6/5.2,<br>3.3/5.5 | 3.8, 3.6, 3.7 | The comparison of two different volumes at the two largest lattice spacings indicates that FV effects are below the statistical errors. No explicit discussion of FSE in ETM 21B. |

Table 108: Finite-volume effects in $N_f = 2+1+1$ determinations of the $D$- and $D_s$-meson decay constants. Each $L$-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, the lightest masses are quoted.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\min}L$ | Description |
|---|---|---|---|---|---|
| RQCD/ALPHA 24 | [13] | 2+1 | 2.34, 2.05/5.46, 2.4/4.8, 2.0/6.1, 2.4/4.8, 2.48 | 4, 3.6, 3.8, 4.05, 4.2, 4.2 | By comparing different volumes (up to 5) at fixed pion mass, FSE are estimated to be negligible once the cut $m_\pi L \geq 3.5$ or $L \geq 2.3$ fm is imposed. |
| ALPHA 23 | [5] | 2+1 | 2.7, 2.4, 3.1/4.1, 2.4/3.1 | 3.9, 5.1, 4.2, 4.1 | No explicit discussion of FSE. |

Table 109: Finite-volume effects in $N_f = 2+1$ determinations of the $D$- and $D_s$-meson decay constants. Each $L$-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, the lightest masses are quoted.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Continuum extrapolation | Scale Setting |
|---|---|---|---|---|---|
| ETM 13F<br>ETM 14E<br>ETM 21B | [10–12] | 2+1+1 | 0.095, 0.081, 0.069 | Chiral and continuum extrapolations performed simultaneously by adding an $O(a^2)$ term to the chiral fits. | Relative scale set through $w_0$ or $M_{c's'}$, the mass of a fictitious meson made of valence quarks of mass $r_0 m_{s'} = 0.22$ and $r_0 m_{c'} = 2.4$. Absolute scale through $f_\pi^{iso}$. |

Table 110: Lattice spacings and description of actions used in $N_f = 2+1+1$ determinations of the $D$- and $D_s$-meson decay constants.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Continuum extrapolation | Scale Setting |
|---|---|---|---|---|---|
| RQCD/ALPHA 24 | [13] | 2+1 | 0.098, 0.085, 0.075, 0.064, 0.049, 0.039 | Terms up to $a^4$ (possibly with mass-dependent coefficients) are included in the chiral-continuum extrapolation. | $t_0$ is used as intermediate scale with the the physical scale set by a combination of $f_\pi$ and $f_K$ in the isosymmetric limit. |
| ALPHA 23 | [5] | 2+1 | 0.085, 0.075, 0.064, 0.049 | $\mathcal{O}(a^2)$ terms, with mass-dependent coefficients, are included in the chiral-continuum extrapolation. | $t_0$ is used as intermediate scale with the the physical scale set by a combination of $f_\pi$ and $f_K$ in the isosymmetric limit. |

Table 111: Lattice spacings and description of actions used in $N_f = 2+1$ determinations of the $D$- and $D_s$-meson decay constants.

| Collab. | Ref. | $N_f$ | Ren. | Description |
|---|---|---|---|---|
| ETM 13F<br>ETM 14E<br>ETM 21B | [10–12] | 2+1+1 | – | The axial current is absolutely normalized. |

Table 112: Operator renormalization in $N_f = 2{+}1{+}1$ determinations of the $D$- and $D_s$-meson decay constants.

| Collab. | Ref. | $N_f$ | Ren. | Description |
|---|---|---|---|---|
| RQCD/ALPHA 24 | [13] | 2+1 | SF | The axial current is nonperturbatively improved and renormalized. |
| ALPHA 23 | [5] | 2+1 | – | The axial current is absolutely normalized. |

Table 113: Operator renormalization in $N_f = 2 + 1$ determinations of the $D$- and $D_s$-meson decay constants.

| Collab. | Ref. | $N_f$ | Action | Description |
|---|---|---|---|---|
| ETM 13F<br>ETM 14E<br>ETM 21B | [10–12] | 2+1+1 | tmWil | $0.15 \lesssim am_h \lesssim 0.28$. |

Table 114: Heavy-quark treatment in $N_f = 2 + 1 + 1$ determinations of the $D$-and $D_s$-meson decay constants.

| Collab. | Ref. | $N_f$ | Action | Description |
|---|---|---|---|---|
| RQCD/ALPHA 24 | [13] | 2+1 | npSW | $0.1 \leq am_h \leq 0.3$. |
| ALPHA 23 | [5] | 2+1 | tmWil on npSW | $0.13 \leq am_h \leq 0.26$ |

Table 115: Heavy-quark treatment in $N_f = 2 + 1$ determinations of the $D$- and $D_s$-meson decay constants.

#### C.4.1 Form factors for semileptonic decays of charmed hadrons

| Collab. | Ref. | $N_f$ | $a$ [fm] | Continuum extrapolation | Scale setting |
|---|---|---|---|---|---|
| FNAL/MILC 22 | [14] | 2+1+1 | 0.12, 0.088, 0.057, 0.042 | Combined chiral-continuum extrapolation using SU(2) heavy-meson rooted staggered chiral perturbation theory. | Scale setting using gradient flow $w_0$ with physical scale from $f_\pi$. |
| Meinel 21B | [15] | 2+1 | 0.0828(3), 0.1106(3) | Combined chiral-continuum extrapolation as part of the expansion of form factor shape in powers of $w-1$. Systematics estimated by varying fit form. | Scale setting using $\Omega$ mass in Ref. [16]. |
| HPQCD 21A | [17] | 2+1+1 | 0.042, 0.06, 0.09, 0.12, 0.15 | Modified $z$-expansion fit combining the continuum and chiral extrapolations and the momentum-transfer dependence. Discretization effects assumed dominated by the charm scale. Discretization errors on form factors between 0.4% and 1.2% as a function of the momentum transfer. | Scale setting from $f_\pi$ via the flow quantity $w_0$ [18–20]. |
| Zhang 21 | [21] | 2+1 | 0.080, 0.11 | Continuum extrapolation combined with fit to $q^2$-dependence of form factors in a "modified" $z$-expansion. Systematics estimated from difference between extrapolated results and results at smallest lattice spacing, and difference between two current renormalization methods. | Set from Wilson-flow quantity $w_0$. |
| HPQCD 20 | [22] | 2+1+1 | 0.06, 0.09, 0.12, 0.15 | Modified $z$-expansion fit combining the continuum and chiral extrapolations and the momentum-transfer dependence, and, for the heavy-HISQ spectator $b$ quark, the dependence on $1/m_Q$. The analysis combines data with NRQCD $b$ quarks and data with HISQ heavy quarks. | Scale setting from $f_\pi$ via the flow quantity $w_0$ [18–20]. |

Table 116: Continuum extrapolations/estimation of lattice artifacts in $N_f = 2+1+1$ determinations of form factors for semileptonic decays of charmed hadrons. For HPQCD 22, see Tab. 142.

| Collab. | Ref. | $N_f$ | $M_{\pi,\mathrm{min}}$ [MeV] | Description |
|---|---|---|---|---|
| FNAL/MILC 22 | [14] | 2+1+1 | 135, 130, 134, 308 | Combined chiral-continuum extrapolation using SU(2) heavy-meson rooted staggered chiral perturbation theory at NLO, including NNLO analytic terms. |
| Meinel 21B | [15] | 2+1 | 303, 340 | Combined chiral-continuum extrapolation as part of the expansion of form factor shape in powers of $w-1$. Systematic uncertainty estimated by repeating fit with added higher-order terms. |
| HPQCD 21A | [17] | 2+1+1 | 315, 329, 129, 132, 131 | Modified $z$-expansion fit combining the continuum and chiral extrapolations and the momentum-transfer dependence. Polynomial dependence on quark masses, supplemented by a pion chiral logarithm. Fit result compared with alternative approach based on cubic splines in $q^2$. |
| Zhang 21 | [21] | 2+1 | 300, 290 | Dependence on pion mass neglected. No estimate of resulting systematic uncertainty. |
| HPQCD 20 | [22] | 2+1+1 | 329, 316, 132/305, 131/305 | Modified $z$-expansion fit combining the continuum and chiral extrapolations and the momentum-transfer dependence, and, for the heavy-HISQ spectator $b$ quark, the dependence on $1/m_Q$. The analysis combines data with NRQCD $b$ quarks and data with HISQ heavy quarks. |

Table 117: Chiral extrapolation/minimum pion mass in determinations of form factors for semileptonic decays of charmed hadrons. For actions with multiple species of pions, masses quoted are the RMS pion masses for $N_f = 2{+}1$ and the Goldstone mode mass for $N_f = 2{+}1{+}1$. The different $M_{\pi,\mathrm{min}}$ entries correspond to the different lattice spacings. For HPQCD 22, see Tab. 143.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\min}L$ | Description |
|---|---|---|---|---|---|
| FNAL/MILC 22 | [14] | 2+1+1 | 5.76, 4.22/5.63, 2.74/3.65/5.47, 2.69 | 3.95, 3.72, 3.72, 4.20 | Finite-volume effects removed by correction to chiral logs due to sums over discrete momenta; corrections are $\mathcal{O}(0.01)\%$ overall. Effect of frozen topological charge at finest lattice spacing also corrected using $\chi$PT and found to be $\lesssim 0.03\%$. |
| Meinel 21B | [15] | 2+1 | 2.7, 2.7 | 4.1, 4.6 | Finite-volume effects not quantified. Effects from unstable $\Lambda^*(1520)$ not quantified. |
| HPQCD 21A | [17] | 2+1+1 | 2.73, 2.72, 2.81/5.62, 2.93/5.87, 2.45/4.89 | $\gtrsim 3.7$ | Finite-volume correction included in chiral fit, claimed to be a negligible effect. Effect of frozen topology in finest ensemble not discussed. |
| Zhang 21 | [21] | 2+1+1 | 2.6, 2.6 | $\gtrsim 3.8$ | No discussion of finite-volume effects. |
| HPQCD 20 | [22] | 2+1+1 | 2.72, 2.81, 2.93/5.87, 2.45/4.89 | $\gtrsim 3.8$ | Physical point ensemble at $a \simeq 0.15$ fm has $m_\pi L = 3.3$; the statement $m_\pi L \gtrsim 3.8$ applies to the other five ensembles. |

Table 118: Finite-volume effects in determinations of form factors for semileptonic decays of charmed hadrons. Each $L$-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, the lightest pion masses are quoted. For HPQCD 22, see Tab. 144.

| Collab. | Ref. | $N_f$ | Ren. | Description |
|---|---|---|---|---|
| FNAL/MILC 22 | [14] | 2+1+1 | NPR | Nonperturbative renormalization by imposing the PCVC relation. |
| Meinel 21B | [15] | 2+1 | mNPR | Residual matching factors $\rho$ computed at 1-loop for vector and axial-vector currents, but at tree-level only for tensor currents. A systematic uncertainty is assigned to $\rho_{T^{\mu\nu}}$ as the double of $\max(\lvert\rho_{A^\mu}-1\rvert, \lvert\rho_{V^\mu}-1\rvert)$. |
| HPQCD 21A | [17] | 2+1+1 | NP | Vector current normalized by imposing Ward identity at zero recoil. |
| Zhang 21 | [21] | 2+1 | NP | Local vector current renormalized using ratio to conserved vector current. Axial current renormalized using ratio of off-shell quark matrix elements. |
| HPQCD 20 | [22] | 2+1+1 | NP | Vector current normalized by imposing Ward identity at zero recoil. |

Table 119: Operator renormalization in determinations of form factors for semileptonic decays of charmed hadrons. For HPQCD 22, see Tab. 145.

| Collab. | Ref. | $N_f$ | Action | Description |
|---|---|---|---|---|
| FNAL/MILC 22 | [14] | 2+1+1 | HISQ | Valence heavy-quark masses range from 0.9 to 2 times the physical charm mass, with $0.164 \leq am_h \leq 0.8935$ |
| Meinel 21B | [15] | 2+1 | Columbia RHQ for both the $b$ and $c$ quarks. | Discretization errors discussed as part of combined chiral-continuum-$w$ fit. Higher-order fit also includes $\mathcal{O}(\alpha_s a\lvert\mathbf{p}\rvert)$ terms to account for missing radiative corrections to $\mathcal{O}(a)$ improvement of the currents. |
| HPQCD 21A | [17] | 2+1+1 | HISQ | Bare charm-quark mass $0.194 \lesssim am_c \lesssim 0.8605$. |
| Zhang 21 | [21] | 2+1+1 | SW | Bare charm-quark mass $0.235 \lesssim am_c \lesssim 0.485$. No $\mathcal{O}(a)$ improvement of currents. |
| HPQCD 20 | [22] | 2+1+1 | Charm: HISQ Bottom (spectator): HISQ and NRQCD | Bare charm-quark HIQS mass $0.274 \lesssim am_c \lesssim 0.827$. Bare bottom-quark HIQS mass $0.274 \lesssim am_b \lesssim 0.8$. |

Table 120: Heavy-quark treatment in determinations of form factors for semileptonic decays of charmed hadrons. For HPQCD 22, see Tab. 146.

## C.5 Notes to Sec. 8 on $B$-meson decay constants, mixing parameters and form factors

### C.5.1 $B_{(s)}$-meson decay constants

| Collab. | Ref. | $N_f$ | $M_{\pi,\rm min}$ [MeV] | Description |
|---|---|---|---|---|
| Frezzotti 24 | [23] | 2+1+1 | 175, 140, 137, 141 | One light-quark mass per lattice spacing. Chiral effects expected to be subdominant compared to other effects. |

Table 121: Chiral extrapolation/minimum pion mass in determinations of the $B$- and $B_s$-meson decay constants for $N_f = 2+1+1$ simulations. The different $M_{\pi,\rm min}$ entries correspond to the different lattice spacings.

| Collab. | Ref. | $N_f$ | $M_{\pi,\rm min}$ [MeV] | Description |
|---|---|---|---|---|
| QCDSF/UKQCD /CSSM 22 | [24] | 2+1 | 280, 155, 226, 290 | Between one and three light-quark masses per lattice spacing. Generic fits to $(M_\pi^2/X_\pi^2 - 1)^2$ and $a^2(M_\pi^2/X_\pi^2 - 1)$ in the combined chiral-continuum extrapolation, with systematic errors estimated to be from 1.3% in $f_{B_s}/f_B$. |
| RBC/UKQCD 22 | [25] | 2+1 | 340, 302, 267, 371 | Between one and three light-quark masses per lattice spacing. Combined chiral-continuum extrapolation using NLO SU(2) Heavy-Meson $\chi$PT. No explicit estimate of systematic errors. |

Table 122: Chiral extrapolation/minimum pion mass in determinations of the $B$- and $B_s$-meson decay constants for $N_f = 2+1$ simulations. The different $M_{\pi,\rm min}$ entries correspond to the different lattice spacings.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\rm min}L$ | Description |
|---|---|---|---|---|---|
| Frezzotti 24 | [23] | 2+1+1 | 4.36, 5.09, 5.46, 5.46 | 3.9, 3.6, 3.8, 3.9 | Finite-volume effects estimated to be subdominant to other sources of uncertainty, based in part on calculations in a larger ensemble in [26] |

Table 123: Finite-volume effects in determinations of the $B$- and $B_s$-meson decay constants for $N_f = 2+1+1$ simulations. Each $L$-entry corresponds to a different lattice spacing.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\min}L$ | Description |
|---|---|---|---|---|---|
| QCDSF/UKQCD /CSSM 22 | [24] | 2+1 | 2.62, 2.36/3.55, 3.26/4.35, 2.83 | 3.86, 3.10/4.07, 4.37/3.42, 4.03 | Final result for $f_{B_s}/f_B$ includes fits to ensembles with $M_\pi L > 4$. No explicit estimate of FV effects. |
| RBC/UKQCD 22 | [25] | 2+1 | 2.65, 2.65, 3.40, 2.00 | 4.57, 4.06, 4.60, 3.77 | No explicit estimate of FV effects. |

Table 124: Finite-volume effects in determinations of the $B$- and $B_s$-meson decay constants for $N_f = 2+1$ simulations. Each $L$-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Continuum extrapolation | Scale setting |
|---|---|---|---|---|---|
| Frezzotti 24 | [23] | 2+1+1 | 0.091, 0.080, 0.068, 0.057 | Continuum extrapolation linear in $a^2$. | Scale set by $w_0$, with details described in Ref. [27]. |

Table 125: Continuum extrapolations/estimation of lattice artifacts in determinations of the $B$- and $B_s$-meson decay constants for $N_f = 2+1+1$ simulations.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Continuum extrapolation | Scale setting |
|---|---|---|---|---|---|
| QCDSF/UKQCD /CSSM 22 | [24] | 2+1 | 0.082, 0.074, 0.068, 0.059 | Combined continuum and chiral extrapolation omits the term linear in $a^2$. Systematic errors associated with discretization effects subdominant in $f_{B_s}/f_B$. | Scale setting procedure and scale uncertainty are not discussed. |
| RBC/UKQCD 22 | [25] | 2+1 | 0.11, 0.083, 0.071, 0.063 | Combined continuum and chiral extrapolation includes term linear in $a^2$. No estimate of systematic errors associated with discretization effects. | Scale setting procedure and scale uncertainty are not discussed. |

Table 126: Continuum extrapolations/estimation of lattice artifacts in determinations of the $B$ and $B_s$ meson decay constants for $N_f = 2+1$ simulations.

| Collab. | Ref. | $N_f$ | Ren. | Description |
|---|---|---|---|---|
| Frezzotti 24 | [23] | 2+1+1 | – | Nonperturbative operator renormalization provided by ETMC by private communication and unpublished at the time of this review. |

Table 127: Description of the renormalization/matching procedure adopted in the determinations of the $B$- and $B_s$-meson decay constants for $N_f = 2+1+1$ simulations.

| Collab. | Ref. | $N_f$ | Ren. | Description |
|---|---|---|---|---|
| QCDSF/UKQCD /CSSM 22 | [24] | 2+1 | mNPR | Operator renormalization is calculated partially non-perturbatively as $Z_{B_q} = \rho_A^{bq}\sqrt{Z_V^{bb}Z_V^{qq}}$, with perturbative contribution neglected, $\rho_A^{bq} = 1$. |
| RBC/UKQCD 22 | [25] | 2+1 | mNPR | Operator renormalization is calculated partially non-perturbatively as $Z_{B_q} = \rho_A^{bq}\sqrt{Z_V^{bb}Z_V^{qq}}$. |

Table 128: Description of the renormalization/matching procedure adopted in the determinations of the $B$- and $B_s$-meson decay constants for $N_f = 2+1$ simulations.

| Collab. | Ref. | $N_f$ | Action | Description |
|---|---|---|---|---|
| Frezzotti 24 | [23] | 2+1+1 | tmWil | Heavy-strange meson extrapolated to physical $B_s$ mass using HQET scaling linear in $B/m_{H_s}$, with contributions from QCD-HQET current matching and HQET axial current anomalous dimension at 1-loop. |

Table 129: Heavy-quark treatment in determinations of the $B$- and $B_s$-meson decay constants for $N_f = 2+1+1$ simulations.

| Collab. | Ref. | $N_f$ | Action | Description |
|---|---|---|---|---|
| QCDSF/UKQCD /CSSM 22 | [24] | 2+1 | RHQ | HQ tuning effects are estimated to be 0.06% in$f_{B_s}/f_B$. HQ discretization effects not explicitly estimated, although the continuum-limit fits do not indicate a strong $a^2$ dependence. |
| RBC/UKQCD 22 | [25] | 2+1 | RHQ | HQ tuning and discretization effects not explicitly estimated. HQ tuning of new finest ensemble ongoing. |

Table 130: Heavy-quark treatment in $N_f = 2+1$ determinations of the $B$-and $B_s$-meson decay constants.

### C.5.2 $B_{(s)}$-meson mixing matrix elements

| Collab. | Ref. | $N_f$ | $a$ [fm] | Continuum extrapolation | Scale setting |
|---|---|---|---|---|---|
| HPQCD 19A | [28] | 2+1+1 | 0.15, 0.12, 0.09 | Discretization errors start from $\alpha_s a^2$ and are included in the systematic error. It is estimated as 1.8% for individual bag parameters. Residual $\alpha_s a^2$ and $a^4$ errors from wrong-spin contributions are subtracted by including them in the chiral fit. | Scale setting done using $\Upsilon$ and $\Upsilon'$ mass splitting [29]. |

Table 131: Continuum extrapolations/estimation of lattice artifacts in determinations of the neutral $B$-meson mixing matrix elements for $N_f = 2+1+1$ simulations.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Continuum extrapolation | Scale setting |
|---|---|---|---|---|---|
| RBC/UKQCD 18A | [30] | 2+1 | 0.11, 0.08, 0.07 | Combined continuum ($a^2$) and heavy quark ($1/m_H$) extrapolation with the LO pion mass dependence ($m_\pi^2$) in the global fit. | Lattice scale and target quark masses are set using $\Omega$, $K$ and $\pi$ masses [16, 31, 32]. |

Table 132: Continuum extrapolations/estimation of lattice artifacts in determinations of the neutral $B$-meson mixing matrix elements for $N_f = 2+1$ simulations.

| Collab. | Ref. | $N_f$ | $M_{\pi,\min}$ [MeV] | Description |
|---|---|---|---|---|
| HPQCD 19A | [28] | 2+1+1 | 311, 241, – | Pion mass in the Goldston channel is as small as 130 MeV for two coarser lattices. NLO HMrS$\chi$PT is used with NNLO analytic terms and other discretization errors. Staggered wrong-spin contributions are included. |
| RBC/UKQCD 18A | [30] | 2+1 | 139, 139, 234 | Combined continuum ($a^2$) and heavy quark ($1/m_H$) extrapolation with the LO pion mass dependence ($m_\pi^2$) in the global fit. |

Table 133: Chiral extrapolation/minimum pion mass in determinations of the neutral $B$-meson mixing matrix elements. For actions with multiple species of pions, masses quoted are the RMS pion masses (where available). The different $M_{\pi,\min}$ entries correspond to the different lattice spacings.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\min}L$ | Description |
|---|---|---|---|---|---|
| HPQCD 19A | [28] | 2+1+1 | 2.4/3.5/4.6, 2.9/3.8/5.7, 2.8 | 7.3, 7.0, - | FV error is estimated to be negligible from FV HM$\chi$PT. |
| RBC/UKQCD 18A | [30] | 2+1 | 2.7/5.5, 2.6/5.3, 3.5 | 3.9, 3.8, 4.0 | FV error is estimated to be less than 0.18% for SU(3)-breaking ratios from FV HM$\chi$PT. |

Table 134: Finite-volume effects in determinations of the neutral $B$-meson mixing matrix elements. Each $L$-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, masses quoted are the RMS pion masses (where available).

| Collab. | Ref. | $N_f$ | Ren. | Description |
|---|---|---|---|---|
| HPQCD 19A | [28] | 2+1+1 | PT1$\ell$ | HISQ-NRQCD 4-quark operators are matched through $O(1/M)$ and renormalized to 1-loop: included are those of $O(\alpha_s)$, $O(\Lambda_{\rm QCD}/M)$, $O(\alpha_s/aM)$, $O(\alpha_s\,\Lambda_{\rm QCD}/M)$. Remnant error is dominated by $O(\alpha_s\Lambda_{\rm QCD}/M)$ 2.9% and $O(\alpha_s^2)$ 2.1% for individual bag parameters. Associated error for their SU(3) breaking ratio are negligible. |
| RBC/UKQCD 18A | [30] | 2+1 | - | Operators are renormalized multiplicatively due to chiral symmetry of DWF. No need to calculate the renormalization factor since only the SU(3) breaking ratios are examined. |

Table 135: Operator renormalization in determinations of the neutral $B$-meson mixing matrix elements.

| Collab. | Ref. | $N_f$ | Action | Description |
|---|---|---|---|---|
| HPQCD 19A | [28] | 2+1+1 | NRQCD | See the entry in Tab. 135. |
| RBC/UKQCD 18A | [30] | 2+1 | DWF | Domain-wall fermion with 3 stout-smearing extends the reach to heavy mass, allowing to simulate up to half of the b-quark mass. Heavy mass errors on $\xi$ are estimated as 0.8% from fitting range and 0.4% from higher order $(1/M^2)$ by power counting. |

Table 136: Heavy-quark treatment in determinations of the neutral $B$-meson mixing matrix elements.

### C.5.3 Form factors entering determinations of $|V_{ub}|$ ($B \to \pi\ell\nu$, $B_s \to K\ell\nu$, $\Lambda_b \to p\ell\bar{\nu}$)

| Collab. | Ref. | $N_f$ | $a$ [fm] | Continuum extrapolation | Scale setting |
|---|---|---|---|---|---|
| RBC/UKQCD 23 | [33] | 2+1 | 0.071, 0.083, 0.11 | Joint chiral-continuum extrapolation using SU(2) hard-pion HM$\chi$PT. Systematic uncertainty estimated by varying fit ansatz and form of coefficients, as well as implementing different cuts on data. | Scale implicitly set in the light-quark sector using the $\Omega^-$ mass, cf. [16, 30, 32]. |
| JLQCD 22 | [34] | 2+1 | 0.044, 0.055, 0.080 | Discretization effects treated using overall factors of $(1 + C_{a^2}(\Lambda_{\rm QCD}a)^2 + C_{(am_Q)^2}(am_Q)^2)$, with independent coefficients for the two form factors. Systematic uncertainties estimated by adding $C_{a^4}(\Lambda_{\rm QCD}a)^4$ or $C_{(am_Q)^4}(am_Q)^4)$ terms. | Relative scale set using gradient-flow time $t_0^{1/2}/a$. Absolute scale $t_0^{1/2}$ taken from Ref. [35]. |
| FNAL/MILC 19 | [36] | 2+1 | 0.06, 0.09, 0.12 | HMrS$\chi$PT expansion used at next-to-leading order in SU(2) and leading order in $1/M_B$, including next-to-next-to-leading-order (NNLO) analytic and generic discretization terms. Hard kaons assumed to decouple. Systematic uncertainties estimated by varying fit ansatz and data range. The (stat + chiral extrap + HQ discretization + $g_\pi$) uncertainty dominates the error budget, ranging from 2–3% at $q^2 \gtrsim 21$ GeV$^2$ to up to 8-10% in the lower end of the accessed $q^2$ interval. | Relative scale $r_1/a$ set from the static-quark potential. Absolute scale $r_1$, including related uncertainty estimates, taken from [37]. |

Table 137: Continuum extrapolations/estimation of lattice artifacts in determinations of $B \to \pi\ell\nu$, $B_s \to K\ell\nu$, and $\Lambda_b \to p\ell\bar{\nu}$ form factors.

| Collab. | Ref. | $N_f$ | $M_{\pi,\text{min}}$ [MeV] | Description |
|---|---|---|---|---|
| RBC/UKQCD 23 | [33] | 2+1 | 268, 301, 340 | Joint chiral-continuum extrapolation using SU(2) hard-pion HM$\chi$PT. Systematic uncertainty estimated by varying fit ansatz and form of coefficients, as well as implementing different cuts on data. |
| JLQCD 22 | [34] | 2+1 | 300, 300, 230 | Chiral extrapolation uses SU(2) hard-pion heavy-meson chiral perturbation theory at next-to-leading order. Systematic uncertainty estimated by adding $M_\pi^4$ terms or by making the coefficients of the chiral logs fit parameters. |
| FNAL/MILC 19 | [36] | 2+1 | 255, 277, 456 | HMrS$\chi$PT expansion used at next-to-leading order in SU(2) and leading order in $1/M_B$, including next-to-next-to-leading-order (NNLO) analytic and generic discretization terms. Hard kaons assumed to decouple. Systematic uncertainties estimated by varying fit ansatz and data range. |

Table 138: Chiral extrapolation/minimum pion mass in determinations of $B \to \pi\ell\nu$, $B_s \to K\ell\nu$, and $\Lambda_b \to p\ell\bar{\nu}$ form factors. For actions with multiple species of pions, masses quoted are the RMS pion masses. The different $M_{\pi,\text{min}}$ entries correspond to the different lattice spacings.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\rm min}L$ | Description |
|---|---|---|---|---|---|
| RBC/UKQCD 23 | [33] | 2+1 | 3.4, 2.7, 2.6 | 4.6, 4.0, 4.4 | Finite-volume effects removed by correction to chiral logs due to sums over discrete momenta; quoted maximum corrections are 0.13% for $f_+$ and 0.06% for $f_0$. |
| JLQCD 22 | [34] | 2+1 | 2.6, 3.9 | $\gtrsim 4.0$ | Finite-volume effects in form factors deemed negligible. Bias in pion mass due to topology freezing at finest lattice spacing estimated to be $\sim 0.1\%$. |
| FNAL/MILC 19 | [36] | 2+1 | 3.8, 2.5/2.9/3.6/5.8, 2.9 | $\gtrsim 3.8$ | Finite-volume effects estimated by comparing infinite volume integrals with finite sums in HMrS$\chi$PT, found to be negligible. |

Table 139: Finite-volume effects in determinations of $B \to \pi\ell\nu$, $B_s \to K\ell\nu$, and $\Lambda_b \to p\ell\bar{\nu}$ form factors. Each $L$-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, the lightest masses are quoted.

| Collab. | Ref. | $N_f$ | Ren. | Description |
|---|---|---|---|---|
| RBC/UKQCD 23 | [33] | 2+1 | mNPR | Perturbative truncation error estimated as full size of $O(\alpha_s)$ correction at the 0.083 fm lattice spacing. |
| JLQCD 22 | [34] | 2+1 | NPR | $Z_{V_{qq}}$ obtained using position-space current-current correlators. For heavier quark masses, $\sqrt{Z_{V_{QQ}}Z_{V_{qq}}}$ is used, where $Z_{V_{QQ}}$ is the renormalization factor of the flavour-conserving temporal vector current, determined using charge conservation. |
| FNAL/MILC 19 | [36] | 2+1 | mNPR | Perturbative truncation error estimated at 1% with size of 1-loop correction on next-to-finest ensemble. |

Table 140: Operator renormalization in determinations of $B \to \pi\ell\nu$, $B_s \to K\ell\nu$, and $\Lambda_b \to p\ell\bar{\nu}$ form factors.

| Collab. | Ref. | $N_f$ | Action | Description |
|---|---|---|---|---|
| RBC/UKQCD 23 | [33] | 2+1 | Columbia RHQ | Heavy-quark discretization errors estimated by power counting. |
| JLQCD 22 | [34] | 2+1 | DWF | Bare heavy-quark masses satisfy $am_Q < 0.7$ and reach from the charm mass up to 2.44 times the charm mass. Form factors extrapolated linearly in $1/m_Q$ to the bottom mass. |
| FNAL/MILC 19 | [36] | 2+1 | Fermilab | (See comments for continuum limit extrapolation.) |

Table 141: Heavy-quark treatment in determinations of $B \to \pi\ell\nu$, $B_s \to K\ell\nu$, and $\Lambda_b \to p\ell\bar{\nu}$ form factors.

### C.5.4 Form factors for rare decays of beauty hadrons

| Collab. | Ref. | $N_f$ | $a$ [fm] | Continuum extrapolation | Scale setting |
|---|---|---|---|---|---|
| HPQCD 22 | [38] | 2+1+1 | 0.15, 0.12, 0.090, 0.088, 0.059, 0.044 | Combined extrapolation in lattice spacing, light-quark mass, strange-quark mass, heavy-quark mass, and momentum transfer using modified $z$ expansion. Stability tested by varying fit form, changing prior widths, and removing data subsets. | Scale setting using gradient flow $w_0$ with physical scale from $f_\pi$. |
| Meinel 20, Meinel 21B | [15, 39] | 2+1 | 0.0828(3), 0.1106(3) | Combined chiral-continuum extrapolation as part of the expansion of form factor shape in powers of $w-1$. Systematic uncertainty estimated by repeating fit with added higher-order terms. | Scale setting using $\Omega$ mass in Ref. [16]. |

Table 142: Continuum extrapolations/estimation of lattice artifacts in determinations of form factors for rare decays of beauty hadrons.

| Collab. | Ref. | $N_f$ | $M_{\pi,\text{min}}$ [MeV] | Description |
|---|---|---|---|---|
| HPQCD 22 | [38] | 2+1+1 | 131, 132, 313, 128, 325, 308 | Combined extrapolation in lattice spacing, light-quark mass, strange-quark mass, heavy-quark mass, and momentum transfer using modified $z$ expansion. Logarithms from hard-pion $\chi$PT included. Stability tested by varying fit form, changing prior widths, and removing data subsets. |
| Meinel 20, Meinel 21B | [15, 39] | 2+1 | 303, 340 | Combined chiral-continuum extrapolation as part of the expansion of form factor shape in powers of $w-1$. Systematic uncertainty estimated by repeating fit with added higher-order terms. |

Table 143: Chiral extrapolation/minimum pion mass in determinations of form factors for rare decays of beauty hadrons. For actions with multiple species of pions, masses quoted are the RMS pion masses for $N_f = 2+1$ and the Goldstone mode mass for $N_f = 2+1+1$. The different $M_{\pi,\text{min}}$ entries correspond to the different lattice spacings.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\min}L$ | Description |
|---|---|---|---|---|---|
| HPQCD 22 | [38] | 2+1+1 | 2.4/4.8, 2.88/5.76, 2.88, 5.63, 2.83, 2.82 | 3.19, 3.86, 4.57, 3.66, 4.67, 4.41 | Finite-volume effects included in fit by replacing infinite-volume chiral logs with sums over discrete momenta. |
| Meinel 20, Meinel 21B | [15, 39] | 2+1 | 2.7, 2.7 | 4.1, 4.6 | Finite-volume effects not quantified. Effects from unstable $\Lambda^*(1520)$ not quantified. |

Table 144: Finite-volume effects in determinations of form factors for rare decays of beauty hadrons. Each $L$-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, masses quoted are the RMS pion masses for $N_f = 2+1$ and the Goldstone mode mass for $N_f = 2+1+1$.

| Collab. | Ref. | $N_f$ | Ren. | Description |
|---|---|---|---|---|
| HPQCD 22 | [38] | 2+1+1 | NPR | $Z_V$ and $Z_A$ obtained from Ward identities. $Z_T$ determined using RI-SMOM. |
| Meinel 20, Meinel 21B | [15, 39] | 2+1 | mNPR | Residual matching factors $\rho$ computed at 1-loop for vector and axial-vector currents, but at tree-level only for tensor currents. A systematic uncertainty is assigned to $\rho_{T^{\mu\nu}}$ as the double of $\max(\lvert\rho_{A^\mu}-1\rvert, \lvert\rho_{V^\mu}-1\rvert)$. |

Table 145: Operator renormalization in determinations of form factors for rare decays of beauty hadrons.

| Collab. | Ref. | $N_f$ | Action | Description |
|---|---|---|---|---|
| HPQCD 22 | [38] | 2+1+1 | HISQ | Extrapolation to the physical $b$-quark mass using terms with powers and logarithms of the inverse heavy-meson mass in the modified $z$-expansion fit. Heavy-quark masses in lattice units satisfy $am_h \leq 0.9$. The heavy-light pseudoscalar meson mass reaches $\approx 0.94\, M_{B,\mathrm{phys.}}$. |
| Meinel 20, Meinel 21B | [15, 39] | 2+1 | Columbia RHQ | Discretization errors discussed as part of combined chiral-continuum-$w$ fit. Higher-order fit also includes $\mathcal{O}(\alpha_s a\lvert\mathbf{p}\rvert)$ terms to account for missing radiative corrections to $\mathcal{O}(a)$ improvement of the currents. |

Table 146: Heavy-quark treatment in determinations of form factors for rare decays of beauty hadrons.

#### C.5.5 Form factors entering determinations of $|V_{cb}|$ ($B_{(s)} \to D^{(*)}_{(s)}\ell\nu$, $\Lambda_b \to \Lambda_c^{(*)}\ell\bar{\nu}$) and $R(D_{(s)})$

| Collab. | Ref. | $N_f$ | $a$ [fm] | Continuum extrapolation | Scale setting |
|---|---|---|---|---|---|
| HPQCD 23 | [40] | 2+1+1 | 0.044, 0.058, 0.088 | Combined chiral-continuum and heavy-quark extrapolations using HMrS$\chi$PT. The recoil dependence in powers of $(w-1)$ is fitted using BGL-inspired coefficients. Zero-recoil uncertainty negligible compared to other sources of error. | Scale setting from Wilson flow, fixing the slope $t\frac{d}{dt}\{t^2\langle E(t)\rangle\}\|_{t=w_0^2} = 0.3$, with $w_0$ taken from [18]. Uncertainty related to scale setting estimated at $\approx 0.5\%$. |
| JLQCD 23 | [41] | 2+1 | 0.044, 0.055, 0.080 | Combined chiral-continuum and heavy-quark extrapolations using HM$\chi$PT. Each form factor is extrapolated separatedly. Zero-recoil uncertainty estimated at $\approx 0.9\%$. | Scale setting from Wilson flow, fixing the slope $t\frac{d}{dt}\{t^2\langle E(t)\rangle\}\|_{t=w_0^2} = 0.3$, with $w_0$ taken from [35]. Uncertainty related to scale setting estimated at $\approx 1.7\%$. |
| FNAL/MILC 21 | [42] | 2+1 | 0.045, 0.06, 0.09, 0.12, 0.15 | Combined chiral-continuum extrapolation using HMrS$\chi$PT. Total uncertainty quoted at 0.7%. | Relative scale $r_1/a$ set from the static-quark potential. Absolute scale $r_1$, including related uncertainty estimates, taken from [37]. Uncertainty related to scale setting estimated at less than 0.1%. |
| Meinel 21, Meinel 21B | [15, 43] | 2+1 | 0.0828(3), 0.1106(3) | Combined chiral-continuum extrapolation as part of the expansion of form factor shape in powers of $w-1$. Systematics estimated by varying fit form. | Scale setting using $\Omega$ mass in Ref. [16]. |

Table 147: Continuum extrapolations/estimation of lattice artifacts in $N_f = 2+1$ determinations of $B_{(s)} \to D^{(*)}_{(s)}\ell\nu$ and $\Lambda_b \to \Lambda_c^{(*)}\ell\bar{\nu}$ form factors, and of $R(D_{(s)})$.

| Collab. | Ref. | $N_f$ | $M_{\pi,\mathrm{min}}$ [MeV] | Description |
|---|---|---|---|---|
| HPQCD 23 | [40] | 2+1+1 | 315, 135, 129 | Combined chiral-continuum and heavy-quark extrapolations using HMrS$\chi$PT. The recoil dependence in powers of $(w-1)$ is fitted using BGL-inspired coefficients. Zero-recoil uncertainty negligible compared to other sources of error. |
| JLQCD 23 | [41] | 2+1 | 284, 300, 226 | Combined chiral-continuum and heavy-quark extrapolations using HM$\chi$PT. Each form factor is extrapolated separately. Zero-recoil uncertainty estimated at $\approx 0.9\%$. |
| FNAL/MILC 21 | [42] | 2+1 | 320, 220, 180, 270, 340 | Combined chiral-continuum extrapolation using HMrS$\chi$PT. Systematic errors estimated by adding higher-order analytic terms and varying the $D^*$-$D$-$\pi$ coupling. Total uncertainty quoted at 0.7%. |
| Meinel 21, Meinel 21B | [15, 43] | 2+1 | 303, 340 | Combined chiral-continuum extrapolation as part of the expansion of form factor shape in powers of $w-1$. Systematic uncertainty estimated by repeating fit with added higher-order terms. |

Table 148: Chiral extrapolation/minimum pion mass in $N_f = 2+1$ determinations of $B_{(s)} \to D^{(*)}_{(s)}\ell\nu$ and $\Lambda_b \to \Lambda_c^{(*)}\ell\bar{\nu}$ form factors, and of $R(D_{(s)})$. For actions with multiple species of pions, masses quoted are the RMS pion masses for $N_f = 2+1$ and the Goldstone mode mass for $N_f = 2+1+1$. The different $M_{\pi,\mathrm{min}}$ entries correspond to the different lattice spacings.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\mathrm{min}}L$ | Description |
|---|---|---|---|---|---|
| HPQCD 23 | [40] | 2+1+1 | 2.8, 2.8–5.5, 2.8–5.6 | 4.5, 3.7, 3.7 | Finite-volume effects expected to be negligible, including the effect of frozen topology in the finest ensemble, according to [14]. |
| JLQCD 23 | [41] | 2+1 | 2.8, 2.6, 2.6–3.9 | 4.0, 4.0, 4.4 | Study effects of topology freezing and compare ensembles with similar parameters but different volumes. |
| FNAL/MILC 21 | [42] | 2+1 | 4.6, 4.3–6.3, 4.1–5.8, 3.8–6.2, 3.9 | $\gtrsim 3.8$ | Finite-volume error estimated to be negligible at zero recoil using HMrS$\chi$PT. Given the values $m_\pi L \gtrsim 3.7$ and the smallness of the chiral logs, expectations are that finite-volume errors remain negligible in the whole recoil range. |
| Meinel 21, Meinel 21B | [15, 43] | 2+1 | 2.7, 2.7 | 4.1, 4.6 | Finite-volume effects not quantified. Effects from unstable $\Lambda_c^*$ not quantified. |

Table 149: Finite-volume effects in determinations of $B_{(s)} \to D^{(*)}_{(s)}\ell\nu$ and $\Lambda_b \to \Lambda_c^{(*)}\ell\bar{\nu}$ form factors, and of $R(D_{(s)})$. Each $L$-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, the lightest pion masses are quoted.

| Collab. | Ref. | $N_f$ | Ren. | Description |
|---|---|---|---|---|
| HPQCD 23 | [40] | 2+1+1 | NPR | Vector (axial) currents renormalized non-perturbatively using the PCVC (PCAC) relation. |
| JLQCD 23 | [41] | 2+1 | mNPR | Majority of current renormalization factor cancels in ratio of lattice correlation functions. Remaining correction expected to behave better than $O(a)$, and vanishes in the continuum limit. |
| FNAL/MILC 21 | [42] | 2+1 | mNPR | Majority of current renormalization factor cancels in double ratio of lattice correlation functions. Remaining correction calculated with 1-loop tadpole-improved lattice perturbation theory. Systematic uncertainty estimated at 0.1%. |
| Meinel 21, Meinel 21B | [15, 43] | 2+1 | mNPR | Residual matching factors $\rho$ computed at 1-loop for vector and axial-vector currents, but at tree-level only for tensor currents. A systematic uncertainty is assigned to $\rho_{T^{\mu\nu}}$ as the double of $\max(\lvert\rho_{A^\mu}-1\rvert, \lvert\rho_{V^\mu}-1\rvert)$. |

Table 150: Operator renormalization in determinations of $B_{(s)} \to D^{(*)}_{(s)}\ell\nu$ and $\Lambda_b \to \Lambda_c^{(*)}\ell\bar{\nu}$ form factors, and of $R(D_{(s)})$.

| Collab. | Ref. | $N_f$ | Action | Description |
|---|---|---|---|---|
| HPQCD 23 | [40] | 2+1+1 | HISQ for both the $b$ and $c$ quarks. | Values of bare heavy-quark masses up to $am_h = 0.8$. The error from continuum limit and extrapolation to physical $b$ mass at zero recoil is quite small, but it becomes dominant at mid-recoil. |
| JLQCD 23 | [41] | 2+1 | Möbius Domain-Wall for both the $b$ and $c$ quarks. | Values of bare heavy-quark masses up to $am_h = 0.69$. The systematics associated to the extrapolation to physical $b$ mass stays under 4% for all form factors in the whole recoil range. |
| FNAL/MILC 21 | [42] | 2+1 | Fermilab RHQ for both the $b$ and $c$ quarks. | Discretization errors discussed as part of combined chiral-continuum stemming from $\alpha_s a$, $a^2$ and $a^3$ terms. |
| Meinel 21, Meinel 21B | [15, 43] | 2+1 | Columbia RHQ for both the $b$ and $c$ quarks. | Discretization errors discussed as part of combined chiral-continuum-$w$ fit. Higher-order fit also includes $\mathcal{O}(\alpha_s a|\mathbf{p}|)$ terms to account for missing radiative corrections to $\mathcal{O}(a)$ improvement of the currents. |

Table 151: Heavy-quark treatment in determinations of $B_{(s)} \to D_{(s)}^{(*)}\ell\nu$ and $\Lambda_b \to \Lambda_c^{(*)}\ell\bar{\nu}$ form factors, and of $R(D_{(s)})$.

## C.6 Notes to Sec. 9 on the strong coupling $\alpha_s$

### C.6.1 Renormalization scale and perturbative behaviour

| Collab. | Ref. | $N_f$ | $\alpha_{\rm eff}$ | $n_l$ | Description |
|---|---|---|---|---|---|
| Hasenfratz 23 | [44] | 0 | $0.095 < \alpha_{\rm eff} < 1.26$ | 2 | GF scheme does not reach perturbative asymptotics. |
| Wong 23 | [45] | 0 | $0.095 < \alpha_{\rm eff} < 1.26$ | 2 | GF scheme does not reach perturbative asymptotics. |
| Brambilla 23 | [46] | 0 | $0.27 < \alpha_{\rm eff} < 0.36$ | 3 | Static force using operator insertion. |
| Chimirri 23 | [47] | 0 | $0.17 < \alpha_{\overline{\rm MS}} < 0.25$ | 2 | Values for $\alpha$ read off from figure. |
| Bribian 21 | [48] | 0 | SF: 0.07–0.19<br>TGF: 0.05–0.92 | 2 | Step scaling with TGF, nonpert. matching to SF. |

Table 152: Renormalization scale and perturbative behaviour of $\alpha_s$ determinations for $N_f = 0$.

| Collab. | Ref. | $N_f$ | $\alpha_{\rm eff}$ | $n_l$ | Description |
|---|---|---|---|---|---|
| ALPHA 22 | [49] | 2+1, 0 | 0.08–0.95 | 2 | Decoupling $N_f = 3 \to N_f = 0$;<br>uses $N_f = 0$ step-scaling from Dalla Brida 19 [50]. |

Table 153: Renormalization scale and perturbative behaviour of $\alpha_s$ determinations for $N_f = 3$.

### C.6.2 Continuum limit

| Collab. | Ref. | $N_f$ | $a\,\mu$ | Description |
|---|---|---|---|---|
| Hasenfratz 23 | [44] | 0 | $0.158 < a/\sqrt{8t} < 0.29$ | GF scheme, infinite-volume extrapolation, direct determination of the $\beta$-function. |
| Wong 23 | [45] | 0 | $0.16 < a/\sqrt{8t} < 0.28$ | GF scheme, infinite-volume extrapolation, direct determination of the $\beta$-function. |
| Brambilla 23 | [46] | 0 | $0.23 < a\mu < 0.49$ | Force between static quarks using operator insertion. |
| Chimirri 23 | [47] | 0 | $0.8m_c < \mu < 3.5m_c$ | Lattice spacings $a$ in the range 0.01–0.07 fm;<br>Scale defined by $\mu = s \times \overline{m}_{\overline{\rm MS},c}(\mu)$. |
| Bribian 21 | [48] | 0 | TGF: $0.041 < a\mu < 0.083$<br>SF: $0.063 < a\mu < 0.17$ | Step scaling TGF scheme,<br>nonpert. matching to SF scheme. |

Table 154: Continuum limit for $\alpha_s$ determinations with $N_f = 0$.

| Collab. | Ref. | $N_f$ | $a\,\mu$ | Description |
|---|---|---|---|---|
| ALPHA 22 | [49] | 2+1, 0 | $0.021 < a\mu_{\rm dec} < 0.083$ | Decoupling $N_f = 3 \to N_f = 0$;<br>continuum limit subject to cutoff $aM < 0.4$,<br>$z = M/\mu_{\rm dec} = 4 - 12$. |

Table 155: Continuum limit for $\alpha_s$ determinations with $N_f = 3$.

### C.7 Notes to Sec. 10 on nucleon matrix elements

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| ETM 23 | [51] | 2+1+1 | 0.057, 0.069, 0.080 | Extrapolation via a fit which is linear in $a^2$. |
| PNDME 23 | [52] | 2+1+1 | 0.06, 0.09, 0.12, 0.15 | Physical-point extrapolations performed simultaneously, keeping only the leading-order terms in the various expansion parameters. |
| ETM 22 | [53] | 2+1+1 | 0.057, 0.069, 0.080 | Extrapolation via a fit which is linear in $a^2$. |

Table 156: Continuum extrapolations/estimation of lattice artifacts in determinations of the isovector axial, scalar and tensor charges with $N_f = 2 + 1 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| Mainz 24 | [54] | 2+1 | 0.049,0.064,0.076,0.086 | Extrapolation performed as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |
| PACS 23 | [55] | 2+1 | 0.063, 0.085 | Discretization effects estimated by difference between two ensembles. |
| RQCD 23 | [56] | 2+1 | 0.039, 0.049, 0.064, 0.076, 0.086, 0.098 | Extrapolation performed using terms up to $a^2$ in the lattice spacing. |
| QCDSF/ UKQCD/ CSSM 23 | [57] | 2+1 | 0.052, 0.059, 0.068, 0.074, 0.082 | Extrapolation performed including leading discretization effects. |
| PACS 22B | [58] | 2+1 | 0.085 | Single lattice spacing. |
| Mainz 22 | [59] | 2+1 | 0.049,0.064,0.076,0.086 | Extrapolation performed as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |

Table 157: Continuum extrapolations/estimation of lattice artifacts in determinations of the isovector axial, scalar and tensor charges with $N_f = 2 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | $M_{\pi,\text{min}}$ [MeV] | Description |
|---|---|---|---|---|
| ETM 23 | [51] | 2+1+1 | 140, 138, 141 | Three pion masses within 3% of the physical value. |
| PNDME 23 | [52] | 2+1+1 | 321, 228, 138, 136 | Fit performed including leading-order pion-mass dependence. |
| ETM 22 | [53] | 2+1+1 | 140, 138, 141 | Three pion masses within 3% of the physical value. |

Table 158: Chiral extrapolation/minimum pion mass in determinations of the isovector axial, scalar and tensor charges with $N_f = 2+1+1$ quark flavours.

| Collab. | Ref. | $N_f$ | $M_{\pi,\text{min}}$ [MeV] | Description |
|---|---|---|---|---|
| Mainz 24 | [54] | 2+1 | 176, 130,218,228 | Physical-point extrapolations were performed simultaneously in the lattice spacing, pion mass, and volume. |
| PACS 23 | [55] | 2+1 | 138, 135 | Three near-physical pion masses. |
| RQCD 23 | [56] | 2+1 | 336,176, 131,156,127,338 | Extrapolations performed using leading-order chiral expressions for the pion mass. |
| QCDSF/ UKQCD/ CSSM 23 | [57] | 2+1 | 290,315,270,220,280,347 | Combined pion-mass, lattice-spacing, and volume extrapolations, performed around chiral SU(3) point. |
| PACS 22B | [58] | 2+1 | 135 | Two near-physical pion masses. |
| Mainz 22 | [59] | 2+1 | 176, 130,218,228 | Physical-point extrapolations were performed simultaneously in the lattice spacing, pion mass, and volume. |

Table 159: Chiral extrapolation/minimum pion mass in determinations of the isovector axial, scalar and tensor charges with $N_f = 2+1$ quark flavours.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\min}L$ | Description |
|---|---|---|---|---|---|
| ETM 23 | [51] | 2+1+1 | 5.1,<br>5.4,<br>5.5 | 3.6,<br>3.8,<br>3.9 | No extrapolation performed. |
| PNDME 23 | [52] | 2+1+1 | 2.4,<br>2.9-4.8,<br>2.9-5.8,<br>2.9-5.8 | 3.62,<br>2.98 | Physical-point extrapolations performed simultaneously, using the leading-order terms in the various expansion parameters. |
| ETM 22 | [53] | 2+1+1 | 5.1,<br>5.4,<br>5.5 | 3.62,<br>2.98 | No extrapolation performed. |

Table 160: Finite-volume effects in determinations of the isovector axial, scalar and tensor charges with $N_f = 2 + 1 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\min}L$ | Description |
|---|---|---|---|---|---|
| Mainz 24 | [54] | 2+1 | 4.1<br>4.8,<br>6.1,<br>4.7 | 4.7,<br>5.4,<br>4.0,<br>4.2 | Extrapolation performed including a term of the form $M_\pi^2 e^{-M_\pi L}/\sqrt{M_\pi L}$ as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |
| PACS 23 | [55] | 2+1 | 5.5-10.9,<br>10.9 | 3.8-7.5,<br>7.5 | Negligible finite-volume effects seen between volumes. |
| RQCD 23 | [56] | 2+1 | 4.8,<br>2.75-11,<br>4.9-9.7,<br>4.1-12.3,<br>6.3-9.4,<br>7.5 | 4.0,<br>3.5,<br>3.8,<br>4.1,<br>4.3,<br>4.3 | Leading asymptotic form for finite-volume corrections used for extrapolation. |
| QCDSF/ UKQCD/ CSSM 23 | [57] | 2+1 | 2.6,<br>2.4,<br>3.3-4.4,<br>2.8,<br>2.5 | 4.0,<br>3.9 | Combined pion-mass, lattice-spacing, and volume extrapolations performed. |
| PACS 22B | [58] | 2+1 | 5.5-10.9 | 3.8-7.5 | Negligible finite-volume effects seen between volumes. |
| Mainz 22 | [59] | 2+1 | 4.1<br>4.8,<br>6.1,<br>4.7 | 4.7,<br>5.4,<br>4.0,<br>4.2 | Extrapolation performed including a term of the form $M_\pi^2 e^{-M_\pi L}/\sqrt{M_\pi L}$ as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |

Table 161: Finite-volume effects in determinations of the isovector axial, scalar and tensor charges with $N_f = 2 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | Ren. |
|---|---|---|---|
| ETM 23 | [51] | 2+1+1 | RI'-MOM |
| PNDME 23 | [52] | 2+1+1 | RI-SMOM |
| ETM 22 | [53] | 2+1+1 | RI'-MOM |

Table 162: Renormalization in determinations of the isovector axial, scalar and tensor charges with $N_f = 2 + 1 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | Ren. |
|---|---|---|---|
| Mainz 24 | [54] | 2+1 | RI-SMOM |
| PACS 23 | [55] | 2+1 | RI-SMOM |
| RQCD 23 | [56] | 2+1 | RI'-SMOM |
| QCDSF/ UKQCD/ CSSM 23 | [57] | 2+1 | RI'-MOM |
| PACS 22B | [58] | 2+1 | RI-SMOM |
| Mainz 22 | [59] | 2+1 | RI-SMOM |

Table 163: Renormalization in determinations of the isovector axial, scalar and tensor charges with $2 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | $\tau$ [fm] | Description |
|---|---|---|---|---|
| ETM 23 | [51] | 2+1+1 | [0.6–1.6]<br>[0.6–1.5]<br>[0.5–1.2] | Compared results from the plateau, summation method and two-state fits. |
| PNDME 23 | [52] | 2+1+1 | [0.8–1.4]<br>[1–1.4,1–1.4,1–1.4,1–1.7]<br>[0.9–1.4,0.9–1.4,0.7–1.4]<br>[1–1.4,1.1–1.4,1–1.4,1.1–1.4,1–1.3] | Several strategies to remove excited-state contributions, including removing $N\pi$ contributions. |
| ETM 22 | [53] | 2+1+1 | [0.6–1.6]<br>[0.6–1.5] | Compared results from the plateau, summation method and two-state fits. |

Table 164: Control of excited-state contamination in determinations of the isovector axial, scalar and tensor charges with $N_f = 2 + 1 + 1$ quark flavours. The comma-separated list of numbers in square brackets denote the range of source-sink separations $\tau$ (in fermi) at each value of the bare coupling.

| Collab. | Ref. | $N_f$ | $\tau$ [fm] | Description |
|---|---|---|---|---|
| Mainz 24 | [54] | 2+1 | [0.4–1.5,0.4–1.5,0.4–1.5,0.4–1.5]<br>[1–1.2]<br>[0.3–1.5,0.3–1.5,0.3–1.5]<br>[0.3–1.4,0.3–1.4,0.3–1.4,0.3–1.4,0.3–1.4]<br>[0.2–1.4,0.2–1.4,0.2–1.4]<br>[0.3–1.4]<br>[0.9–1.5] | Two-state fits to the summation method. |
| PACS 23 | [55] | 2+1 | [0.8–1.1]<br>[0.75–1.3,0.75–1.7]<br>[0.72–1.46,0.72–1.46]<br>[0.8–1.4,0.9–1.5] | Excited-state contributions estimated using different time separations and smearings. |
| RQCD 23 | [56] | 2+1 | [0.7–1.3,0.7–1.3,0.7–1.3]<br>[0.5–1.2,0.5–1.2,0.5–1.2,0.5–1.2,0.5–1.2,0.5–1.2,0.5–1.2,0.5–1.2,0.5–1.2,0.5–1.2,0.5–1.2,0.5–1.2,0.5–1.2,0.5–1.2,0.5–1.2]<br>[0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2]<br>[0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2]<br>[0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2,0.7–1.2]<br>[0.7–1.3,0.7–1.3] | Simultaneous two- and three-state fits of up to four different observables using four time separations. |
| QCDSF/ UKQCD/ CSSM 23 | [57] | 2+1 | all | Energies from fits to two-point correlation functions, where a weighted average is taken of the results obtained when varying the fitting range. |

Table 165: Control of excited-state contamination in determinations of the isovector axial, scalar and tensor charges with $N_f = 2+1$ quark flavours. The comma-separated list of numbers in square brackets denote the range of source-sink separations $\tau$ (in fermi) at each value of the bare coupling.

| Collab. | Ref. | $N_f$ | $\tau$ [fm] | Description |
|---|---|---|---|---|
| PACS 22B | [58] | 2+1 | [0.9–1.4] | Excited-state contributions estimated using different time separations and smearings. |
| Mainz 22 | [59] | 2+1 | [0.4–1.5,0.4–1.5,0.4–1.5,0.4–1.5]<br>[1–1.2]<br>[0.3–1.5,0.3–1.5,0.3–1.5]<br>[0.3–1.4,0.3–1.4,0.3–1.4,0.3–1.4,0.3–1.4]<br>[0.2–1.4,0.2–1.4,0.2–1.4]<br>[0.3–1.4]<br>[0.9–1.5] | Two-state fits to the summation method. |

Table 165: (cntd.) Control of excited-state contamination in determinations of the isovector axial, scalar and tensor charges with $N_f = 2 + 1$ quark flavours. The comma-separated list of numbers in square brackets denote the range of source-sink separations $\tau$ (in fermi) at each value of the bare coupling.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| PNDME 21 | [60] | 2+1+1 | 0.12,0.09,0.06 | Joint continuum and chiral fit includes a $aM_\pi^2$ term. |
| Mainz 23 | [61] | 2+1 | 0.08,0.07,0.06,0.05 | Joint continuum, chiral and finite-volume (correlated) fit of $\sigma_{\pi N}$ and $\sigma_s$ includes a $aM_{\pi,K}^2$ term. Fits are performed including and excluding this term. |

Table 166: Continuum extrapolation/estimation of lattice artifacts in direct determinations of $\sigma_{\pi N}$ and $\sigma_s$.

| Collab. | Ref. | $N_f$ | $M_{\pi,\min}$ [MeV] | Description |
|---|---|---|---|---|
| PNDME 21 | [60] | 2+1+1 | 228,138,235 | Joint continuum and chiral fit including the SU(2) NNLO baryon $\chi$PT [62] terms. |
| Mainz 23 | [61] | 2+1 | 219, 154, 128, 174 | Joint continuum, chiral and finite-volume (correlated) fit of $\sigma_{\pi N}$ and $\sigma_s$ utilizing SU(3) NNLO covariant baryon $\chi$PT with the EOMS loop regularization scheme [63–65]. Cuts on the pion mass excluding ensembles with $M_\pi > 220$, 285 or 360 MeV are made. |

Table 167: Chiral extrapolation/minimum pion mass in direct determinations of $\sigma_{\pi N}$ and $\sigma_s$.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\min}L$ | Description |
|---|---|---|---|---|---|
| PNDME 21 | [60] | 2+1+1 | 2.9–3.8<br>2.8–5.6<br>2.8–3.7 | 4.4,<br>3.9<br>4.4 | Finite-volume terms are not included in the joint continuum-chiral extrapolation. |
| Mainz 23 | [61] | 2+1 | 2.7–4.1<br>2.4–4.8<br>2.0–6.1<br>2.4–4.7 | 4.6<br>3.8<br>4.0<br>4.2 | Joint continuum, chiral and finite-volume (correlated) fit of $\sigma_{\pi N}$ and $\sigma_s$ includes a term derived from the SU(2) finite-volume expression for the nucleon mass in Ref. [66]. Fits are performed including and excluding this term. |

Table 168: Finite-volume effects in direct determinations of $\sigma_{\pi N}$ and $\sigma_s$.

| Collab. | Ref. | $N_f$ | Ren. | Description |
|---|---|---|---|---|
| PNDME 21 | [60] | 2+1+1 | -/- | Flavour mixing occurs due to breaking of chiral symmetry. The ratio of $Z^{ns}/Z^s$ is estimated to be close to 1 and the mixing is neglected. |
| Mainz 23 | [61] | 2+1 | NP/NP | Flavour mixing occurs due to breaking of chiral symmetry. The mixing is implemented using ratios of the light- and strange-quark masses rather than utilising $Z^{ns}/Z^s$. |

Table 169: Renormalization for direct determinations of $\sigma_{\pi N}$ and $\sigma_s$. The type of renormalization (Ren.) is given for $\sigma_{\pi N}$ first and $\sigma_s$ second. The label 'na' indicates that no renormalization is required.

| Collab. | Ref. | $N_f$ | $\tau$ [fm] | Description |
|---|---|---|---|---|
| PNDME 21 | [60] | 2+1+1 | [1.0–1.7,1.0–1.7]/all<br>[0.9–1.4,0.9–1.4]/all<br>[1.1–1.4,1.1–1.4]/all | The two- and three-point functions are fitted simultaneously including four and three states, respectively. The final results are obtained using a narrow-width prior to set the first excited-state energy to that of the lowest $N\pi$ state. |
| Mainz 23 | [61] | 2+1 | [0.3–1.5]/all<br>[0.3–1.5]/all<br>[0.3–1.4]/all<br>[0.2–1.4]/all | Summation method including the ground-state terms and ratio fits including an excited state with the energy fixed with a prior to the lowest $N\pi$ energy are considered. The final result combines results from both fit types. |

Table 170: Control of excited-state contamination in direct determinations of $\sigma_{\pi N}$ and $\sigma_s$. The comma-separated list of numbers in square brackets denote the range of source-sink separations $\tau$ (in fermi) at each value of the bare coupling. The range of $\tau$ for the connected (disconnected) contributions to the three-point correlation functions is given first (second). If a wide range of $\tau$ values is available this is indicated by "all" in the table.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| RQCD 22 | [67] | 2+1 | 0.10,0.09,0.08, 0.06,0.05,0.04 | Combined continuum, chiral and volume fit to the baryon octet. Leading $O(a^2)$ terms are included in the parameterisation. Fits are performed also excluding the coarsest lattice spacing. |

Table 171: Continuum extrapolations/estimation of lattice artifacts in determinations of $\sigma_{\pi N}$ and $\sigma_s$ from the Feynman-Hellmann method.

| Collab. | Ref. | $N_f$ | $M_{\pi,\min}$ [MeV] | Description |
|---|---|---|---|---|
| RQCD 22 | [67] | 2+1 | 338,127,216 131,176,336 | Combined continuum, chiral and volume fit to the baryon octet. Fits utilizing SU(3) NNLO covariant baryon $\chi$PT with the EOMS loop regularization scheme [63–65] are performed. Cuts on the flavour average meson mass squared are made. |

Table 172: Chiral extrapolation/minimum pion mass in determinations of $\sigma_{\pi N}$ and $\sigma_s$ from the Feynman-Hellmann method.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\min}L$ | Description |
|---|---|---|---|---|---|
| RQCD 22 | [67] | 2+1 | 2.3, 2.0–5.5, 2.4–4.8, 2.0–6.1, 2.4–4.7 2.5 | 4.0, 3.5, 5.3, 4.0, 4.2, 4.2 | Combined continuum, chiral and volume fit to the baryon octet. Finite-volume terms from NNLO covariant baryon $\chi$PT (with no new fit parameters) are included in the fit [68–70]. Ensembles with $L < 2.3$ fm are excluded from the analysis. |

Table 173: Finite-volume effects in determinations of $\sigma_{\pi N}$ and $\sigma_s$ from the Feynman-Hellmann method.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| ETM 22 | [53] | 2+1+1 | 0.08,0.07,0.06 | Linear extrapolation in $a^2$. |
| PNDME 20A | [71] | 2+1+1 | 0.15,0.12,0.09,0.06 | Extrapolation performed using a linear term in $a$ as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |
| ETM 20C | [72] | 2+1+1 | 0.08 | Single lattice spacing. |
| ETM 19A | [73] | 2+1+1 | 0.08 | Single lattice spacing. |

Table 174: Continuum extrapolations/estimation of lattice artifacts in determinations of the isovector unpolarised, helicity and transversity second moments with $N_f = 2 + 1 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| Mainz 24 | [54] | 2+1 | 0.09,0.08,0.06,0.05 | A number of simultaneous $a$, $M_\pi$ and $M_\pi L$ fits are performed using a linear term in $a$. The final results are obtained from a weighted average. |
| LHPC 24 | [74] | 2+1 | 0.12,0.09 | A Bayesian fit is performed including an $a$ and an $a^2$ term. |
| NME 21A | [75] | 2+1 | 0.127, 0,09, 0.07 | Extrapolation performed using a linear term in $a$ as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |
| NME 20 | [76] | 2+1 | 0.127, 0,09, 0.07 | Extrapolation performed using a linear term in $a$ as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |
| Mainz 19 | [77] | 2+1 | 0.05,0.06,0.08,0.09 | Extrapolation performed as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |
| $\chi$QCD 18A | [78] | 2+1 | 0.143, 0.11, 0.114, 0.083 | Partially quenched calculation. Extrapolation performed as part of a simultaneous fit in $a^2$, $M_\pi$ and $M_\pi L$ using expression from HBChPT. |
| LHPC 12A | [79] | 2+1 | 0.12,0.09 | No statistically significant discretization effects observed. Results assumed to be constant in $a$. |
| LHPC 10 | [80] | 2+1 | 0.12 | Single lattice spacing. |
| RBC/UKQCD 10D | [81] | 2+1 | 0.11 | Single lattice spacing. |

Table 175: Continuum extrapolations/estimation of lattice artifacts in determinations of the isovector unpolarised, helicity and transversity second moments with $N_f = 2 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| RQCD 18 | [82] | 2 | 0.081, 0.071, 0.060 | No significant $O(a)$ effects observed. |
| ETM 17C | [83] | 2 | 0.0938 | Single ensemble. |
| ETM 15D | [84] | 2 | 0.093 | Single ensemble. |
| RQCD 14A | [85] | 2 | 0.081, 0.071, 0.060 | Analysis not conclusive. |

Table 176: Continuum extrapolations/estimation of lattice artifacts in determinations of the isovector unpolarised, helicity and transversity second moments with $N_f = 2$ quark flavours.

| Collab. | Ref. | $N_f$ | $M_{\pi,\text{min}}$ [MeV] | Description |
|---|---|---|---|---|
| ETM 22 | [53] | 2+1+1 | 140,137,141 | Simulate close to $M_\pi^{phys}$. |
| PNDME 20A | [71] | 2+1+1 | 321,228,138,136 | Extrapolation performed using a linear term in $M_\pi^2$ as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |
| ETM 20C | [72] | 2+1+1 | 139 | Single pion mass within 3% of the physical value. |
| ETM 19A | [73] | 2+1+1 | 139 | Single pion mass within 3% of the physical value. |

Table 177: Chiral extrapolation/minimum pion mass in determinations of the isovector unpolarised, helicity and transversity second moments with $N_f = 2 + 1 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | $M_{\pi,\mathrm{min}}$ [MeV] | Description |
|---|---|---|---|---|
| Mainz 24 | [54] | 2+1 | 228,218,130,176 | A number of simultaneous $a$, $M_\pi$ and $M_\pi L$ fits are performed using terms which appear in the SU(2) NNLO ChPT expression for the axial charge and (alternatively) a linear term in $M_\pi^2$. The final results are obtained from a weighted average. |
| LHPC 24 | [74] | 2+1 | 136,133 | Simulate close to $M_\pi^{phys}$. |
| NME 21A | [75] | 2+1 | 285, 169, 167 | Extrapolation performed using a linear term in $M_\pi^2$ as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |
| NME 20 | [76] | 2+1 | 285, 169, 167 | Extrapolation performed using a linear term in $M_\pi^2$ as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |
| Mainz 19 | [77] | 2+1 | 290, 200, 260 | Extrapolation performed using logarithmic and quadratic terms in $M_\pi$ as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |
| $\chi$QCD 18A | [78] | 2+1 | 171,330,139,300 | Partially quenched calculation. Extrapolation performed as part of a simultaneous fit in $a^2$, $M_\pi$ and $M_\pi L$ using expression from HBChPT. |
| LHPC 12A | [79] | 2+1 | 149,317 | Chiral fit formula based on the "small-scale expansion" to order $\epsilon^3$ with some coefficients fixed. |
| LHPC 10 | [80] | 2+1 | 293 | Chiral fit formula based on the "small-scale expansion" to order $\epsilon^3$ with some coefficients fixed. |
| RBC/UKQCD 10D | [81] | 2+1 | 329,416,555,668 | Constant fit to heaviest three and linear fit to lightest two pion masses gives the quoted range. |

Table 178: Chiral extrapolation/minimum pion mass in determinations of the isovector unpolarised, helicity and transversity second moments with $N_f = 2 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | $M_{\pi,\mathrm{min}}$ [MeV] | Description |
|---|---|---|---|---|
| RQCD 18 | [82] | 2 | 280, 150, 260 | Chiral fit using BChPT. Kept terms up to $O(M_\pi^3)$. |
| ETM 17C | [83] | 2 | ≈135 | Single ensemble within 3% of physical pion mass. |
| ETM 15D | [84] | 2 | 131 | Single ensemble within 3% of physical pion mass. |
| RQCD 14A | [85] | 2 | 280, 150, 260 | No significant dependence on pion mass observed. |

Table 179: Chiral extrapolation/minimum pion mass in determinations of the isovector unpolarised, helicity and transversity second moments with $N_f = 2$ quark flavours.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\mathrm{min}}L$ | Description |
|---|---|---|---|---|---|
| ETM 22 | [53] | 2+1+1 | 5.1, 5.5, 5.5 | 3.6, 3.8, 3.9 | Finite-volume effects not estimated. |
| PNDME 20A | [71] | 2+1+1 | 2.4, 4.8, 5.6, 5.5 | 3.9, 5.5, 3.9, 3.7 | Fit performed using a term of the form $M_\pi^2 e^{-M_\pi L}/\sqrt{M_\pi L}$ as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |
| ETM 20C | [72] | 2+1+1 | 5.1 | 3.6 | Finite-volume effects not estimated. |
| ETM 19A | [73] | 2+1+1 | 5.1 | 3.6 | Finite-volume effects not estimated. |

Table 180: Finite-volume effects in determinations of the isovector unpolarised, helicity and transversity second moments with $N_f = 2 + 1 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\min}L$ | Description |
|---|---|---|---|---|---|
| Mainz 24 | [54] | 2+1 | 4.1, 4.8, 6.1, 4.7 | 4.7, 5.4, 4.0, 4.2 | A number of simultaneous $a$, $M_\pi$ and $M_\pi L$ fits are performed using term of the form $M_\pi^2 e^{-M_\pi L}/\sqrt{M_\pi L}$. The final results are obtained from a weighted average. |
| LHPC 24 | [74] | 2+1 | 5.6, 5.9 | 3.9, 4.0 | Finite-volume effects are not estimated. |
| NME 21A | [75] | 2+1 | 4.1, 3.0–5.8, 3.5–5.1 | 5.9, 5.1, 4.3 | Fit performed using a term of the form $M_\pi^2 e^{-M_\pi L}/\sqrt{M_\pi L}$ as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |
| NME 20 | [76] | 2+1 | 4.1, 3.0–5.8, 3.5–5.1 | 5.9, 5.1, 4.3 | Fit performed using a term of the form $M_\pi^2 e^{-M_\pi L}/\sqrt{M_\pi L}$ as part of a simultaneous fit in $a$, $M_\pi$ and $M_\pi L$. |
| Mainz 19 | [77] | 2+1 | 2.8-4.1, 2.4-3.6, 2.1-4.1, 2.4-3.2 | 4.7, 5.3, 4.2, 4.3 | Extrapolation performed including a term of the form $M_\pi^2 e^{-M_\pi L}/\sqrt{M_\pi L}$ as part of a simultaneous fit in $a^2$, $M_\pi$ and $M_\pi L$. |
| $\chi$QCD 18A | [78] | 2+1 | 4.6, 2.7, 5.5, 2.6 | 4.0, 4.4, 3.9, 4.0 | Extrapolation performed including a term of the form $e^{-M_\pi L}$ as part of a simultaneous fit in $a^2$, $M_\pi$ and $M_\pi L$. |
| LHPC 12A | [79] | 2+1 | 2.8–5.6, 2.9 | 4.2, 4.6 | Finite-volume effects investigated and found to be negligible. |
| LHPC 10 | [80] | 2+1 | 2.5–3.5 | 3.7 | Finite-volume effects included in chiral fit formula and found to be negligible. |
| RBC/UKQCD 10D | [81] | 2+1 | 2.7 | 4.6 | No uncertainty estimated. Comparison between two volumes for the three heaviest pion masses shows no deviation within statistical errors. |

Table 181: Finite-volume effects in determinations of the isovector unpolarised, helicity and transversity second moments with $N_f = 2 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | $L$ [fm] | $M_{\pi,\min}L$ | Description |
|---|---|---|---|---|---|
| RQCD 18 | [82] | 2 | 2.6,<br>1.7-4.6,<br>1.9-2.9 | 3.7,<br>2.8,<br>3.8 | No significant finite-volume effects observed. |
| ETM 17C | [83] | 2 | 4.5 | 3.1 | Single ensemble |
| ETM 15D | [84] | 2 | 4.5 | 3.0 | Single ensemble |
| RQCD 14A | [85] | 2 | 2.6,<br>1.7-4.6,<br>1.9-2.9 | 3.7,<br>2.8,<br>3.8 | No significant finite-volume effects observed. |

Table 182: Finite-volume effects in determinations of the isovector unpolarised, helicity and transversity second moments with $N_f = 2$ quark flavours.

| Collab. | Ref. | $N_f$ | Ren. |
|---|---|---|---|
| ETM 22 | [53] | 2+1+1 | RI′-MOM |
| PNDME 20A | [71] | 2+1+1 | RI′-MOM |
| ETM 20C | [72] | 2+1+1 | RI′-MOM |
| ETM 19A | [73] | 2+1+1 | RI′-MOM |
| Mainz 24 | [54] | 2+1 | RI-MOM |
| LHPC 24 | [74] | 2+1 | RI′-MOM/RI-SMOM |
| NME 21A | [75] | 2+1 | RI′-MOM |
| NME 20 | [76] | 2+1 | RI′-MOM |
| Mainz 19 | [77] | 2+1 | RI-MOM |
| $\chi$QCD 18A | [78] | 2+1 | RI-MOM |
| LHPC 12A | [79] | 2+1 | RI-MOM |
| LHPC 10 | [80] | 2+1 | RI-MOM |
| RBC/UKQCD 10D | [81] | 2+1 | RI-MOM |
| RQCD 18 | [82] | 2 | RI′-MOM |
| ETM 17C | [83] | 2 | RI′-MOM |
| ETM 15D | [84] | 2 | RI′-MOM |
| RQCD 14A | [85] | 2 | RI′-MOM |

Table 183: Renormalization in determinations of the isovector unpolarised, helicity and transversity second moments with $N_f = 2+1+1$, $N_f = 2+1$ and $N_f = 2$ quark flavours.

| Collab. | Ref. | $N_f$ | $\tau$ [fm] | Description |
|---|---|---|---|---|
| ETM 22 | [53] | 2+1+1 | [0.6–1.6]<br>[0.6–1.5]<br>[0.5–1.2] | Two-state fit to all $\tau$. A comparison is made with plateau fits and the summation method. |
| PNDME 20A | [71] | 2+1+1 | [0.8–1.4]<br>[1.0–1.7,1.0–1.7]<br>[0.9–1.4,0.9–1.4,0.9–1.4]<br>[1.0–1.4,0.9–1.3] | Fits to the $\tau$- and $t$-dependence of three-point correlators using two or three lowest-lying states. |
| ETM 20C | [72] | 2+1+1 | [0.6–1.6] | Two-state fit to all $\tau$. A comparison is made with plateau fits and the summation method. |
| ETM 19A | [73] | 2+1+1 | [0.6–1.6] | Two-state fit to all $\tau$. A comparison is made with plateau fits and the summation method. |

Table 184: Control of excited-state contamination in determinations of the isovector unpolarised, helicity and transversity second moments with $N_f = 2 + 1 + 1$ quark flavours. The comma-separated list of numbers in square brackets denote the range of source-sink separations $\tau$ (in fermi) at each value of the bare coupling.

| Collab. | Ref. | $N_f$ | $\tau$ [fm] | Description |
|---|---|---|---|---|
| Mainz 24 | [54] | 2+1 | [0.4–1.5,0.4–1.5,0.4–1.5,0.4–1.5] [0.3–1.5,0.3–1.5,0.3–1.5] [0.3–1.4,0.3–1.4,0.3–1.4,0.3–1.4,0.3–1.4] [0.2–1.4,0.2–1.4,0.2–1.4] | Two-state summation method. |
| LHPC 24 | [74] | 2+1 | [0.3–1.4] [0.9–1.5] | Two-state fits to the three-point function and the summation method are utilised, with the final results obtained from a weighted average. |
| NME 21A | [75] | 2+1 | [1.3–1.8] [1.1–1.5,1.3–1.7,1.1–1.5,1.1–1.5] [1.1–1.4,1.2–1.5] | Fits to the $\tau$- and $t$-dependence of three-point correlators using two or three lowest-lying states. |
| NME 20 | [76] | 2+1 | [1.3–1.8] [1.1–1.5,1.3–1.7,1.1–1.5,1.1–1.5] [1.1–1.4,1.2–1.5] | Fits to the $\tau$- and $t$-dependence of three-point correlators using two or three lowest-lying states. |
| Mainz 19 | [77] | 2+1 | [1.0–1.4,1.0–1.4,1.0–1.4] [1.0–1.5,1.0–1.5] [1.0–1.4,1.0–1.4,1.0–1.4,1.0–1.4] [1.0–1.4,1.0–1.3] | Fits to the $\tau$- and $t$-dependence of correlator ratios using the two lowest-lying states. |
| $\chi$QCD 18A | [78] | 2+1 | [0.7–1.5] | two-state fits to the three-point function. |
| LHPC 12A | [79] | 2+1 | [0.9–1.4,0.9–1.4,0.9–1.4,0.9–1.4,0.9–1.4] [0.9–1.4] | Fits to the leading (ground state) $\tau$-dependence of summed correlator ratios. |
| LHPC 10 | [80] | 2+1 | [1.1,1.1,1.1,1.1,1.1–1.2,1.1] | Plateau fits of correlator ratio at $\tau = 1.1\,\mathrm{fm}$. Larger source-sink separation on one ensemble as cross check. |
| RBC/UKQCD 10D | [81] | 2+1 | [1.4,1.4,1.4,1.4] | Single source-sink separation considered. |

Table 185: Control of excited-state contamination in determinations of the isovector unpolarised, helicity and transversity second moments with $N_f = 2 + 1$ quark flavours. The comma-separated list of numbers in square brackets denote the range of source-sink separations $\tau$ (in fermi) at each value of the bare coupling.

| Collab. | Ref. | $N_f$ | $\tau$ [fm] | Description |
|---|---|---|---|---|
| RQCD 18 | [82] | 2 | Not given | Analysis limited by statistics |
| ETM 17C | [83] | 2 | [0.9–1.5] | Result from plateau method with $\tau/a = 14$. Consistent with estimates from summation and two-state fit methods. |
| ETM 15D | [84] | 2 | [0.9–1.3] | Result from plateau method with $\tau/a = 14$. Consistent with estimate from the summation method. |
| RQCD 14A | [85] | 2 | Not given | Plateau value at larger $\tau/a$ consistent with two-state fit. |

Table 186: Control of excited-state contamination in determinations of the isovector unpolarised, helicity and transversity second moments with $N_f = 2$ quark flavours. The comma-separated list of numbers in square brackets denote the range of source-sink separations $\tau$ (in fermi) at each value of the bare coupling.

## C.8 Notes to Sec. 11 on scale setting

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| TUMQCD 22 | [86] | 2+1+1 | 0.15294, 0.12224, 0.08786, 0.05662, 0.0426, 0.03216 | MILC ensembles from on-shell Symanzik-improved gauge action and rooted HISQ fermions |
| ETM 21 | [7] | 2+1+1 | 0.069, 0.079, 0.097 | |

Table 187: Continuum extrapolations/estimation of lattice artifacts in scale determinations with $N_f = 2 + 1 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | $a$ [fm] | Description |
|---|---|---|---|---|
| Hudspith 24 | [87] | 2+1 | 0.098, 0.085, 0.075, 0.064, 0.049, 0.039 | NP $\mathcal{O}(a)$-improved Wilson fermions with tree-level Symanzik improved gauge action. |
| Asmussen 23 | [88] | 2+1 | 0.085, 0.075, 0.064, 0.049, 0.039 | NP $\mathcal{O}(a)$-improved Wilson fermions with tree-level Symanzik improved gauge action. |
| RQCD 22 | [67] | 2+1 | 0.098, 0.085, 0.075, 0.064, 0.049, 0.039 | NP $\mathcal{O}(a)$-improved Wilson fermions with tree-level Symanzik improved gauge action. |
| RBC/Bielefeld 07 | [89] | 2+1 | 0.3 - 0.05 | Simulations with improved staggered fermions (p4fat3-action: smeared 1-link term and bent 3-link terms) at 27 different values of $\beta$. |

Table 188: Continuum extrapolations/estimation of lattice artifacts in scale determinations with $N_f = 2 + 1$ quark flavours.

| Collab. | Ref. | $N_f$ | $M_{\pi,\text{min}}$ [MeV] | $M_\pi L$ | Description |
|---|---|---|---|---|---|
| TUMQCD 22 | [86] | 2+1+1 | 129 | $3.25 - 4.17$ | At the physical point. |
| ETM 21 | [7] | 2+1+1 | 134.2 | 3.78 | |

Table 189: Chiral extrapolation and finite-volume effects in scale determinations with $N_f = 2+1+1$ quark flavours. We list the minimum pion mass $M_{\pi,\text{min}}$ and $M_\pi L \equiv M_{\pi,\text{min}}[L(M_{\pi,\text{min}})]_{\text{max}}$ is evaluated at the maximum value of $L$ available at $M_\pi = M_{\pi,\text{min}}$.

| Collab. | Ref. | $N_f$ | $M_{\pi,\text{min}}$ [MeV] | $M_\pi L$ | Description |
|---|---|---|---|---|---|
| Hudspith 24 | [87] | 2+1 | 127/131 | 3.51/4.05 | At $m = m_{\text{symm}}$. |
| Asmussen 23 | [88] | 2+1 | 127/131 | 3.51/4.05 | At $m = m_{\text{symm}}$. |
| RQCD 22 | [67] | 2+1 | 127/131<br>200 | 3.51/4.05<br>4.14 | At $m = m_{\text{symm}}$.<br>At $\tilde{m}_s = \tilde{m}_{s,\text{phys}}$ |
| RBC/Bielefeld 07 | [89] | 2+1 | 220 | 5.456 | |

Table 190: Chiral extrapolation and finite-volume effects in scale determinations with $N_f = 2+1$ quark flavours. We list the minimum pion mass $M_{\pi,\text{min}}$ and $M_\pi L \equiv M_{\pi,\text{min}}[L(M_{\pi,\text{min}})]_{\text{max}}$ is evaluated at the maximum value of $L$ available at $M_\pi = M_{\pi,\text{min}}$.

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