4 Quark masses

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Quark masses are fundamental parameters of the Standard Model. An accurate determination of these parameters is important for both phenomenological and theoretical applications. The bottom- and charm-quark masses, for instance, are important sources of parametric uncertainties in several Higgs decay modes. The up-, down- and strange-quark masses govern the amount of explicit chiral symmetry breaking in QCD. From a theoretical point of view, the values of quark masses provide information about the flavour structure of physics beyond the Standard Model. The Review of Particle Physics of the Particle Data Group contains a review of quark masses [1], which covers light as well as heavy flavours. Here, we also consider light- and heavy-quark masses, but focus on lattice results and discuss them in more detail. We do not discuss the top quark, however, because it decays weakly before it can hadronize, and the nonperturbative QCD dynamics described by present day lattice calculations is not relevant. The lattice determination of light- (up, down, strange), charm- and bottom-quark masses is considered below in Secs. 4.1, 4.2, and 4.3, respectively.

Quark masses cannot be measured directly in experiment because quarks cannot be isolated, as they are confined inside hadrons. From a theoretical point of view, in QCD with N_f flavours, a precise definition of quark masses requires one to choose a particular renormalization scheme. This renormalization procedure introduces a renormalization scale μ, and quark masses depend on this renormalization scale according to the Renormalization Group (RG) equations. In mass-independent renormalization schemes the RG equations read

$$\mu \frac{\mathrm{d}\bar{m}_i(\mu)}{\mathrm{d}\mu} = \bar{m}_i(\mu)\tau(\bar{g})\,, \tag{28}$$

where the function $\tau(\bar{g})$ is the anomalous dimension, which depends only on the value of the strong coupling $\alpha_s = \bar{g}^2/(4\pi)$. Note that in QCD $\tau(\bar{g})$ is the same for all quark flavours. The anomalous dimension is scheme dependent, but its perturbative expansion

$$\tau(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^2\left(d_0 + d_1\bar{g}^2 + \dots\right) \tag{29}$$

has a leading coefficient $d_0 = 8/(4\pi)^2$, which is scheme independent.[1] Equation (28), being a first order differential equation, can be solved exactly by using Eq. (29) as the boundary condition. The formal solution of the RG equation reads

$$M_i = \bar{m}_i(\mu)[2b_0\bar{g}^2(\mu)]^{-d_0/(2b_0)} \exp\left\{-\int_0^{\bar{g}(\mu)} \mathrm{d}x \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x}\right]\right\}\,, \tag{30}$$

where $b_0 = (11 - 2N_f/3)/(4\pi)^2$ is the universal leading perturbative coefficient in the expansion of the β-function

$$\beta(\bar{g}) \equiv \mu\frac{\mathrm{d}\bar{g}}{\mathrm{d}\mu} \overset{\bar{g}\to 0}{\sim} -\bar{g}^3\left(b_0 + b_1\bar{g}^2 + \dots\right) \tag{31}$$

which governs the running of the strong coupling. The renormalization group invariant (RGI) quark masses M_i are formally integration constants of the RG Eq. (28). They are scale independent, and due to the universality of the coefficient d_0, they are also scheme independent.

[1] We follow the conventions of Gasser and Leutwyler [2].

Moreover, they are nonperturbatively defined by Eq. (30). They only depend on the number of flavours N_f, making them a natural candidate to quote quark masses and compare determinations from different lattice collaborations. Nevertheless, it is customary in the phenomenology community to use the $\overline{\rm MS}$ scheme at a scale $\mu = 2$ GeV to compare different results for light quarks and the charm quark, and to use a scale equal to its own mass for the charm and bottom. In this review, we will quote final averages for both quantities.

Results for quark masses are always quoted in the four-flavour theory unless otherwise noted. $N_f = 2+1$ results have to be converted to the four-flavour theory. Fortunately, the charm quark is heavy $(\Lambda_{\rm QCD}/m_c)^2 < 1$, and this conversion can be performed in perturbation theory with negligible ($\sim 0.2\%$) perturbative uncertainties.

Nonperturbative corrections in this matching are more difficult to estimate. Lattice determinations do not show any significant deviation between $N_f = 2+1$ and $N_f = 2+1+1$ calculations. For example, the difference in the final averages for the mass of the strange quark m_s between $N_f = 2+1$ and $N_f = 2+1+1$ determinations is about 1.3%, or about one standard deviation. Since these effects are suppressed by a factor of $1/N_{\rm c}$, and a factor of the strong coupling at the scale of the charm mass, naive power counting arguments would suggest that the effects are $\sim 1\%$, in line with the above observation. On the other hand, numerical nonperturbative studies [3–5] have found this power counting argument to be an overestimate by one order of magnitude in the determination the Λ-parameter and other quantities.

We quote all final averages at 2 GeV in the $\overline{\rm MS}$ scheme and also the RGI values (in the four-flavour theory). We use the exact RG Eq. (30). Note that to use this equation we need the value of the strong coupling in the $\overline{\rm MS}$ scheme at a scale $\mu = 2$ GeV. All our results are obtained from the RG equation in the $\overline{\rm MS}$ scheme and the 5-loop beta function together with the value of the Λ-parameter in the four-flavour theory $\Lambda^{(4)}_{\overline{\rm MS}} = 295(10)\,\text{MeV}$ obtained in this review (see Sec. 9). We use the 5-loop mass anomalous dimension as well [6]. In the uncertainties of the RGI masses, we separate the contributions from the determination of the quark masses and the propagation of the uncertainty of $\Lambda^{(4)}_{\overline{\rm MS}}$. These are identified with the subscripts m and Λ, respectively.

Conceptually, all lattice determinations of quark masses contain three basic ingredients:

1. Tuning the lattice bare-quark masses to match the experimental values of some quantities. Pseudo-scalar meson masses provide the most common choice, since they have a strong dependence on the values of quark masses.

2. Renormalization of the bare-quark masses. Bare-quark masses determined with the above-mentioned criteria have to be renormalized. Many of the latest determinations use some nonperturbatively defined scheme. One can also use perturbation theory to connect directly the values of the bare-quark masses to the values in the $\overline{\rm MS}$ scheme at 2 GeV. Experience shows that 1-loop calculations are unreliable for the renormalization of quark masses: usually at least two loops are required to have trustworthy results.

3. If quark masses have been nonperturbatively renormalized, for example, to some MOM or SF scheme, the values in this scheme must be converted to the phenomenologically useful values in the $\overline{\rm MS}$ scheme (or to the scheme/scale independent RGI masses). Either option requires the use of perturbation theory. The larger the energy scale of this matching with perturbation theory, the better, and many recent computations in MOM schemes do a nonperturbative running up to 3–4 GeV. Computations in the SF

scheme allow us to perform this running nonperturbatively over large energy scales and match with perturbation theory directly at the electro-weak scale ~ 100 GeV.

Note that many lattice determinations of quark masses make use of perturbation theory at a scale of a few GeV.

We mention that lattice-QCD calculations of the b-quark mass have an additional complication which is not present in the case of the charm and light quarks. At the lattice spacings currently used in numerical calculations the direct treatment of the b quark with the fermionic actions commonly used for light quarks is very challenging. Only two determinations of the b-quark mass use this approach, reaching the physical b-quark mass region at two lattice spacings with $aM \sim 1$. There are a few widely used approaches to treat the b quark on the lattice, which have already been discussed in the FLAG 13 review (see Sec. 8 of Ref. [7]). Those relevant for the determination of the b-quark mass will be briefly described in Sec. 4.3.

4.1 Masses of the light quarks

Light-quark masses are particularly difficult to determine because they are very small (for the up and down quarks) or small (for the strange quark) compared to typical hadronic scales. Thus, their impact on typical hadronic observables is minute, and it is difficult to isolate their contribution accurately.

Fortunately, the spontaneous breaking of $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$ chiral symmetry provides observables which are particularly sensitive to the light-quark masses: the masses of the resulting Nambu-Goldstone bosons (NGB), i.e., pions, kaons, and eta. Indeed, the Gell-Mann-Oakes-Renner relation [8] predicts that the squared mass of a NGB is directly proportional to the sum of the masses of the quark and antiquark which compose it, up to higher-order mass corrections. Moreover, because these NGBs are light, and are composed of only two valence particles, their masses have a particularly clean statistical signal in lattice-QCD calculations. In addition, the experimental uncertainties on these meson masses are negligible. Thus, in lattice calculations, light-quark masses are typically obtained by renormalizing the input quark mass and tuning them to reproduce NGB masses, as described above.

4.1.1 Lattice determination of m_s and m_{ud}

We now turn to a review of the lattice calculations of the light-quark masses and begin with m_s, the isospin-averaged up- and down-quark mass m_{ud}, and their ratio. Most groups quote only m_{ud}, not the individual up- and down-quark masses. We then discuss the ratio m_u/m_d and the individual determinations of m_u and m_d.

Quark masses have been calculated on the lattice since the mid-nineties. However, early calculations were performed in the quenched approximation, leading to unquantifiable systematics. Thus, in the following, we only review modern, unquenched calculations, which include the effects of light sea quarks.

Tables 7 and 8 list the results of $N_f = 2+1$ and $N_f = 2+1+1$ lattice calculations of m_s and m_{ud}. These results are given in the $\overline{\mathrm{MS}}$ scheme at 2 GeV, which is standard nowadays, though some groups are starting to quote results at higher scales (e.g., Ref. [9]). The tables also show the colour coding of the calculations leading to these results. As indicated earlier in this review, we treat calculations with different values of N_f separately.

$N_f = 2+1$ *lattice calculations*

We begin with $N_f = 2+1$ calculations (see FLAG 19 and earlier editions for two-flavour results). These and the corresponding results for m_{ud} and m_s are summarized in Tab. 7. Given the very high precision of a number of the results, with total errors on the order of 1%, it is important to consider the effects neglected in these calculations. Isospin-breaking and electromagnetic effects are small on m_{ud} and m_s, and have been approximately accounted for in the calculations that will be retained for our averages. We have already commented that the effect of the omission of the charm quark in the sea is expected to be small, below our current precision, and we do not add any additional uncertainty due to these effects in the final averages.

The only new computation since the previous FLAG edition is the determination of light-quark masses by the CLQCD collaboration (CLQCD 23 [10]). Using stout-smeared clover fermions, the ensembles reach the physical point and have three lattice spacings to perform the continuum extrapolation. These look under control, having in all cases $\delta(a_{\rm min}) < 2$ (see 2.1.2). Volumes are large, and these characteristics ensure that the rating is ★ in all criteria. Renormalization is performed nonperturbatively in two different setups (RI/MOM and SMOM), with the difference used as a systematic effect. This systematic effect, in fact, dominates their error budget.

The ALPHA collaboration [37] uses nonperturbatively $\mathcal{O}(a)$ improved Wilson fermions (a subset of the CLS ensembles [38]). The renormalization is performed nonperturbatively in the SF scheme from 200 MeV up to the electroweak scale ~ 100 GeV [39]. This nonperturbative running over such large energy scales avoids any use of perturbation theory at low energy scales, but adds a cost in terms of uncertainty: the running alone propagates to $\approx 1\%$ of the error in quark masses. This turns out to be one of the dominant pieces of uncertainty for the case of m_s. On the other hand, for the case of m_{ud}, the uncertainty is dominated by the chiral extrapolations. The ensembles used include four values of the lattice spacing below 0.09 fm, which qualifies for a ★ in the continuum extrapolation, and pion masses down to 200 MeV. This value lies just at the boundary of the ★ rating, but since the chiral extrapolation is a substantial source of systematic uncertainty, we opted to rate the work with a ○. In any case, this work enters in the average and their results show a reasonable agreement with the FLAG average. In all cases the data driven continuum limit criteria shows $\delta(a_{\rm min}) < 3$.

We now comment in some detail on previous works that also contribute to the averages.

RBC/UKQCD 14 [13] significantly improves on their RBC/UKQCD 12B [9] work by adding three new domain wall fermion ensembles to three used previously. Two of the new simulations are performed at essentially physical pion masses ($M_\pi \simeq 139\,\text{MeV}$) on lattices of about 5.4 fm in size and with lattice spacings of 0.114 fm and 0.084 fm. It is complemented by a third simulation with $M_\pi \simeq 371\,\text{MeV}$, $a \simeq 0.063$ fm and a rather small $L \simeq 2.0\,\text{fm}$. Altogether, this gives them six simulations with six unitary ($m_{\rm sea} = m_{\rm val}$) M_π's in the range of 139 to 371 MeV, and effectively three lattice spacings from 0.063 to 0.114 fm. They perform a combined global continuum and chiral fit to all of their results for the π and K masses and decay constants, the Ω baryon mass and two Wilson-flow parameters. Quark masses in these fits are renormalized and run nonperturbatively in the RI-SMOM scheme. This is done by computing the relevant renormalization constant for a reference ensemble, and determining those for other simulations relative to it by adding appropriate parameters in the global fit. This calculation passes all of our selection criteria, with $\delta(a_{\rm min}) \approx 1$.

$N_f = 2+1$ MILC results for light-quark masses go back to 2004 [31, 32]. They use rooted staggered fermions. By 2009 their simulations covered an impressive range of parameter space, with lattice spacings going down to 0.045 fm, and valence-pion masses down to approximately

180 MeV [25]. The most recent MILC $N_f = 2+1$ results, i.e., MILC 10A [19] and MILC 09A [25], feature large statistics and 2-loop renormalization. Since these data sets subsume those of their previous calculations, these latest results are the only ones that need to be kept in any world average.

The BMW 10A, 10B [16, 17] calculation still satisfies our stricter selection criteria. They reach the physical up- and down-quark mass by *interpolation* instead of by extrapolation. Moreover, their calculation was performed at five lattice spacings ranging from 0.054 to 0.116 fm, with small extrapolations $\delta(a_{\rm min}) < 2$. The work uses full nonperturbative renormalization and running and in volumes of up to $(6\ \text{fm})^3$, guaranteeing that the continuum limit, renormalization, and infinite-volume extrapolation are controlled. It does neglect, however, isospin-breaking effects, which are small on the scale of their error bars.

Finally, we come to another calculation which satisfies our selection criteria, HPQCD 10 [20]. It updates the staggered-fermions calculation of HPQCD 09A [24]. In those papers, the renormalized mass of the strange quark is obtained by combining the result of a precise calculation of the renormalized charm-quark mass m_c with the result of a calculation of the quark-mass ratio m_c/m_s. As described in Ref. [36] and in Sec. 4.2, HPQCD determines m_c by fitting Euclidean-time moments of the $\bar{c}c$ pseudoscalar density two-point functions, obtained numerically in lattice QCD, to fourth-order, continuum perturbative expressions. These moments are normalized and chosen so as to require no renormalization with staggered fermions. Since m_c/m_s requires no renormalization either, HPQCD's approach displaces the problem of lattice renormalization in the computation of m_s to one of computing continuum perturbative expressions for the moments. To calculate m_{ud} HPQCD 10 [20] use the MILC 09 determination of the quark-mass ratio m_s/m_{ud} [26].

HPQCD 09A [24] obtains $m_c/m_s = 11.85(16)$ [24] fully nonperturbatively, with a precision slightly larger than 1%. HPQCD 10's determination of the charm-quark mass, $m_c(m_c) = 1.268(6)$,[2] is even more precise, achieving an accuracy better than 0.5%.

This discussion leaves us with six results for our final average for m_s: CLQCD 23 [10], ALPHA 19 [37], MILC 09A [25], BMW 10A, 10B [16, 17], HPQCD 10 [20] and RBC/UKQCD 14 [13]. Assuming that the result from HPQCD 10 is 100% correlated with that of MILC 09A, as it is based on a subset of the MILC 09A configurations, we find $m_s = 92.3(1.0)\,\text{MeV}$ with a $\chi^2/\text{dof} = 1.60$.

For the light-quark mass m_{ud}, the results satisfying our criteria are CLQCD 23, ALPHA 19, RBC/UKQCD 14B, BMW 10A, 10B, HPQCD 10, and MILC 10A. For the error, we include the same 100% correlation between statistical errors for the latter two as for the strange case, resulting in the following (at scale 2 GeV in the $\overline{\rm MS}$ scheme, and χ^2/dof=1.4),

$$N_f = 2+1: \qquad \begin{aligned} m_{ud} &= 3.387(39)\ \text{MeV} \qquad &&\text{Refs. [11, 13, 16, 17, 19, 20]}, \\ m_s &= 92.4(1.0)\ \text{MeV} \qquad &&\text{Refs. [11, 13, 16, 17, 20, 25]}. \end{aligned} \tag{32}$$

And the RGI values

$$N_f = 2+1: \qquad \begin{aligned} M_{ud}^{\rm RGI} &= 4.714(55)_m(46)_\Lambda\ \text{MeV} \qquad &&\text{Refs. [10, 11, 13, 16, 17, 19, 20]}, \\ M_s^{\rm RGI} &= 128.5(1.4)_m(1.2)_\Lambda\ \text{MeV} \qquad &&\text{Refs. [10, 11, 13, 16, 17, 20, 25]}. \end{aligned} \tag{33}$$

$N_f = 2+1+1$ *lattice calculations*

[2] To obtain this number, we have used the conversion from $\mu = 3$ GeV to m_c given in Ref. [36].

Since the previous review a new computation of m_s, m_{ud} has appeared, ETM 21A [40]. Using twisted-mass fermions with an added clover term to suppress $\mathcal{O}(a^2)$ effects between the neutral and charged pions, this work represents a significant improvement over ETM 14 [41]. Renormalization is performed nonperturbatively in the RI-MOM scheme. Their ensembles comprise three lattice spacings (0.095, 0.082, and 0.069 fm), two volumes for the finest lattice spacings with pion masses reaching down to the physical point in the two finest lattices spacings allowing a controlled chiral extrapolation. Their volumes are large, with $m_\pi L$ between four and five. These characteristics of their ensembles pass the most stringent FLAG criteria in all categories. This work extracts quark masses from two different quantities, one based on the meson spectrum and the other based on the baryon spectrum. Results obtained with these two methods agree within errors, but the size of the continuum extrapolation is much larger for the case of the extractions based on the meson spectrum. In particular, we estimate that $\delta(a_{\rm min}) = 4$–4.5 for the individual fits that enter the determination of $m_{\rm ud}, m_{\rm s}$ respectively. We note that while these values are somewhat large, the systematic errors that the authors estimate in the determinations of the light-quark masses are about the same size as the statistical fluctuations. This will reduce the stretching factors to a value close to one, and, therefore we do not apply any additional corrections for these cases. Nevertheless, we stress that some large continuum extrapolations are present in this work.

Determinations based on the baryon spectrum agree well with the FLAG average while the ones based on the meson sector are high in comparison (there is good agreement with their previous results, ETM 14 [41]). Related with the previous point, it is important to note that the determinations that involve large continuum extrapolations are the ones that show a larger tension.

There are three other works that enter in light-quark mass averages. Contributing both to the average of m_{ud} and m_s is FNAL/MILC/TUMQCD 18 [42]. They perform a determination of the strange-quark mass using masses of the heavy-strange mesons as input. In this case, some very large continuum extrapolations, with $\delta(a_{\rm min}) \approx 14$ enter in a global analysis, but for the determination of the light-quark masses, we believe that the influence of the data at heavier masses on the determination of the fit parameter that determines $m_{\rm s}$ is small. In the region $m_{\rm heavy} < 3$ GeV the extrapolations are much better under control, and in fact involve up to five lattice spacing. We conclude that the large value of $\delta(a_{\rm min})$ does not influence significantly the values of the light-quark masses. HPQCD 18 [43] and HPQCD 14A [44] contribute to the determination of m_{ud}, and both show $\delta(a_{\rm min}) < 3$ for most of their region of parameters.

The $N_f = 2+1+1$ results are summarized in Tab. 8. While the results of HPQCD 14A and HPQCD 18 agree well (using different methods), there are several tensions in the determination of m_s. The most significant discrepancy is between the results of the ETM collaboration and other results. But also the two very precise determinations of HPQCD 18 and FNAL/MILC/TUMQCD 18 show a tension. Note that the results of Ref. [44] are reported as $m_s(2\,{\rm GeV}; N_f = 3)$ and those of Ref. [41] as $m_{ud(s)}(2\,{\rm GeV}; N_f = 4)$. We convert the former to $N_f = 4$ and obtain $m_s(2\,{\rm GeV}; N_f = 4) = 93.7(8)$MeV. The average of ETM 21A, FNAL/MILC/TUMQCD 18, HPQCD 18, ETM 14 and HPQCD 14A is 93.46(58)MeV with $\chi^2/{\rm dof} = 1.3$. For the light-quark mass, we average ETM 21A, ETM 14 and FNAL/MILC/TUMQCD 18 to obtain 3.427(51) with a $\chi^2/{\rm dof} = 4.5$. We note these χ^2 values are large. For the case of the light-quark masses there is a clear tension between the ETM and FNAL/MILC/TUMQCD results. We also note that the 2+1-flavour values are consistent with the four-flavour ones, so in all cases we have simply quoted averages according to FLAG rules, including stretching factors for the errors based on χ^2 values of our fits. Nevertheless it is worth pointing out that large continuum extrapolations are present in the $N_f = 2+1+1$ determination of quark masses. Global fits that aim at describing results obtained for a wide range of quark masses are involved in many analyses. At small quark masses many lattice spacing enter these determinations, but how the large quark mass region influences the precision obtained at small quark masses is something that deserves further investigation.

$$N_f = 2+1+1: \qquad \begin{aligned} m_{ud} &= 3.427(51)\ \text{MeV} \qquad \text{Refs. [40–42]}, \\ m_s &= 93.46(58)\ \text{MeV} \qquad \text{Refs. [40–44]}, \end{aligned} \tag{34}$$

and the RGI values

$$N_f = 2+1+1: \qquad \begin{aligned} M_{ud}^{\text{RGI}} &= 4.768(71)_m(46)_\Lambda\ \text{MeV} \qquad \text{Refs. [40–42]}, \\ M_s^{\text{RGI}} &= 130.0(0.8)_m(1.3)_\Lambda\ \text{MeV} \qquad \text{Refs. [40–44]}. \end{aligned} \tag{35}$$

In Figs. 1 and 2 the lattice results listed in Tabs. 7 and 8 and the FLAG averages obtained at each value of N_f are presented and compared with various phenomenological results.

4.1.2 Lattice determinations of m_s/m_{ud}

The lattice results for m_s/m_{ud} are summarized in Tab. 9. In the ratio m_s/m_{ud}, one of the sources of systematic error—the uncertainties in the renormalization factors—drops out. This is especially important for the recent determination by the CLQCD collaboration, since their error budget for the individual quark masses was dominated by the systematic associated with the renormalization. Also, other systematic effects (like the effect of the scale setting) are reduced in these ratios. This might explain that despite the discrepancies that are present in the individual quark mass determinations, the ratios show an overall very good agreement.

$N_f = 2+1$ *lattice calculations*

CLQCD 23 [10], discussed already, is the only new result for this section. The other works contributing to this average are ALPHA 19, RBC/UKQCD 14B, which replaces RBC/UKQCD 12 (see Sec. 4.1.1), and the results of MILC 09A and BMW 10A, 10B.

The results show very good agreement with a $\chi^2/\text{dof} = 0.14$. The final uncertainty ($\approx 0.5\%$) is smaller than the ones of the quark masses themselves. At this level of precision, the uncertainties in the electromagnetic and strong isospin-breaking corrections might not be completely negligible. Nevertheless, we decided not to add any uncertainty associated with this effect. The main reason is that most recent determinations try to estimate this uncertainty themselves and found an effect smaller than naive power counting estimates (see $N_f = 2+1+1$ section),

$$N_f = 2+1: \qquad m_s/m_{ud} = 27.42\ (12) \qquad \text{Refs. [13, 16, 17, 25, 37]}. \tag{36}$$

$N_f = 2+1+1$ *lattice calculations*

For $N_f = 2+1+1$ there are four results, ETM 21 [40], MILC 17 [51], ETM 14 [41] and FNAL/MILC 14A [52], all of which satisfy our selection criteria.

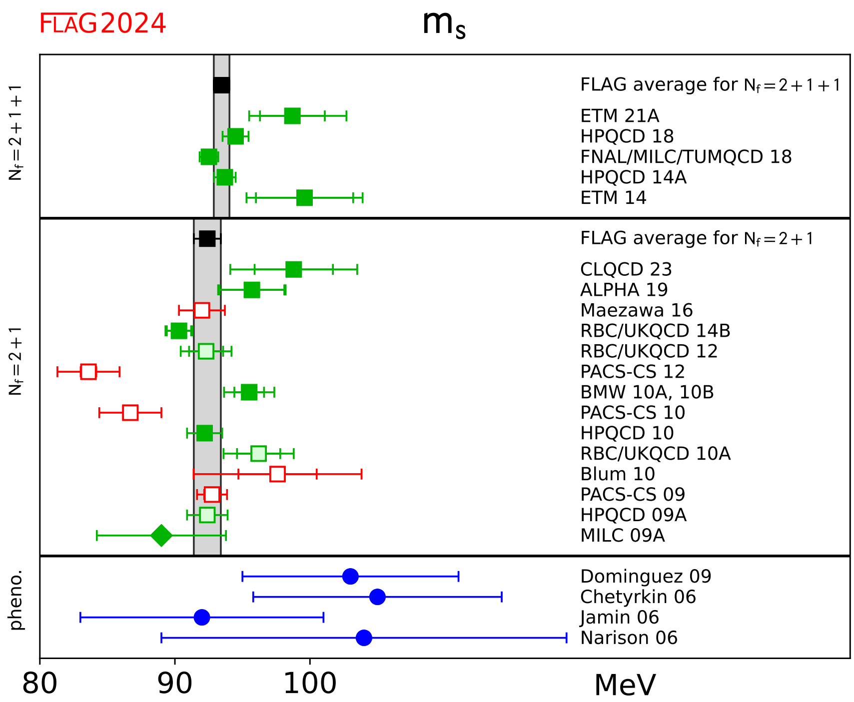

Figure 1: $\overline{\rm MS}$ mass of the strange quark (at 2 GeV scale) in MeV. The upper two panels show the lattice results listed in Tabs. 7 and 8, while the bottom panel collects sum rule results [45–49]. Diamonds and squares represent results based on perturbative and nonperturbative renormalization, respectively. The black squares and the grey bands represent our averages (32) and (34). The significance of the colours is explained in Sec. 2.

All these works have been discussed in the previous FLAG edition [53], except the new result ETM 21A, that we have already examined. The fit has $\chi^2/\mathrm{dof} \approx 1.7$, and the result shows reasonable agreement with the $N_f = 2+1$ result.

$$N_f = 2+1+1: \qquad m_s/m_{ud} = 27.227\ (81) \qquad \text{Refs. [40, 41, 51, 52]}, \tag{37}$$

which corresponds to an overall uncertainty equal to 0.4%. It is worth noting that Ref. [51] estimates the EM effects in this quantity to be $\sim$ 0.18% (or 0.049 which is less than the quoted error above).

All the lattice results listed in Tab. 9 as well as the FLAG averages for each value of N_f are reported in Fig. 3 and compared with χPT and sum rules.

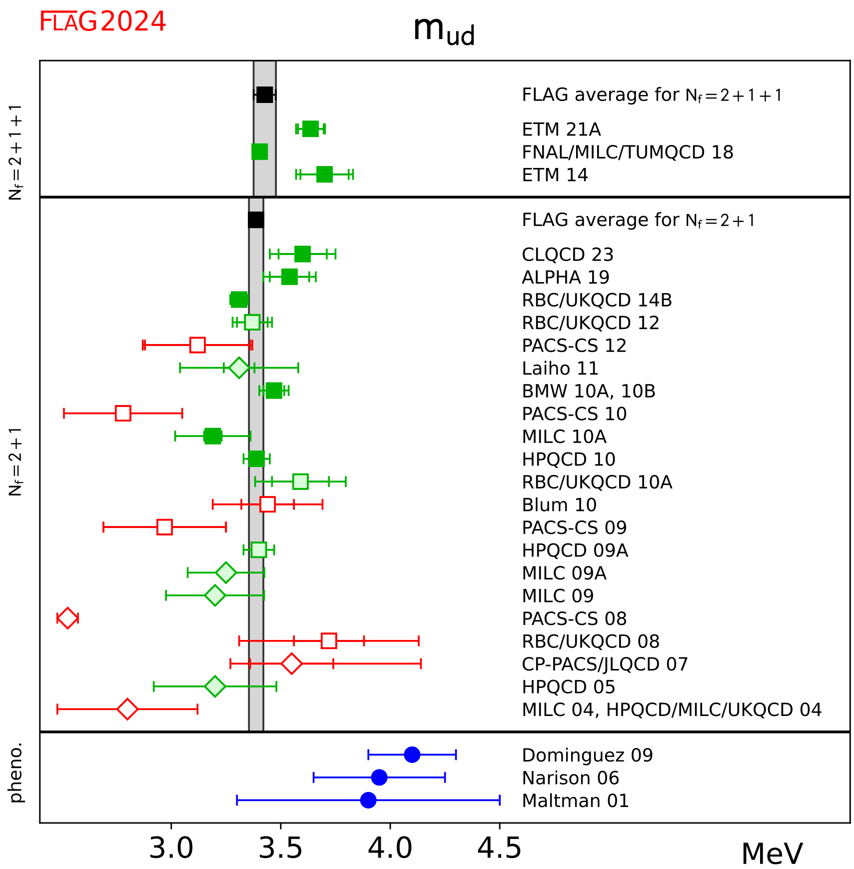

Figure 2: Mean mass of the two lightest quarks, $m_{ud} = \frac{1}{2}(m_u + m_d)$. The bottom panel shows results based on sum rules [45, 48, 50] (for more details see Fig. 1).

4.1.3 Lattice determination of m_u and m_d

In this section, we review computations of the individual m_u and m_d quark masses, as well as the parameter ϵ related to the violations of Dashen's theorem

$$\epsilon = \frac{(\Delta M_K^2 - \Delta M_\pi^2)_\gamma}{\Delta M_\pi^2}, \tag{38}$$

where $\Delta M_\pi^2 = M_{\pi^+}^2 - M_{\pi^0}^2$ and $\Delta M_K^2 = M_{K^+}^2 - M_{K^0}^2$ are the pion and kaon squared mass splittings, respectively. The subscript γ, here and in the following, denotes corrections that arise from electromagnetic effects only according to the prescription given in Section 3. This parameter is often a crucial intermediate quantity in the extraction of the individual light-quark masses. Indeed, it can be shown using the G-parity symmetry of the pion triplet, that ΔM_π^2 does not receive $\mathcal{O}(m_u - m_d)$ isospin-breaking corrections. In other words

$$\Delta M_\pi^2 = (\Delta M_\pi^2)_\gamma \qquad \text{and} \qquad \epsilon = \frac{(\Delta M_K^2)_\gamma}{\Delta M_\pi^2} - 1, \tag{39}$$

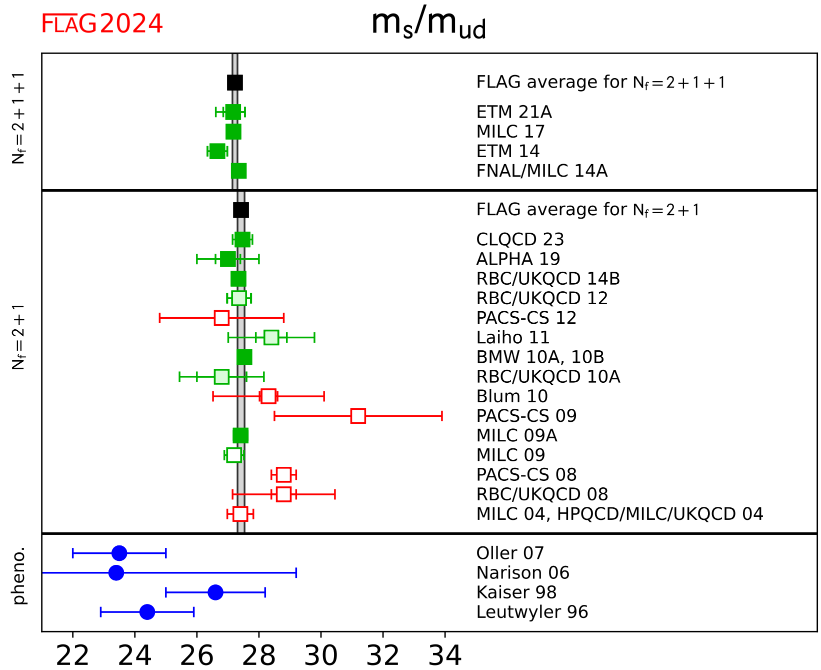

Figure 3: Results for the ratio m_s/m_{ud}. The upper part shows the lattice results listed in Tab. 9 together with the FLAG averages for each value of N_f. The lower part shows results obtained from χPT and sum rules [48, 54–57].

at leading order in the isospin-breaking expansion. Once known, ϵ allows one to consistently subtract the electromagnetic part of the kaon-mass splitting to obtain the QCD part of the kaon mass splitting $(\Delta M_K^2)_{\mathrm{SU(2)}}$. In contrast with the pion, the kaon QCD splitting is sensitive to $m_u - m_d$ and, in particular, proportional to it at leading order in χPT. Therefore, the knowledge of ϵ allows for the determination of $m_u - m_d$ from a chiral fit to lattice-QCD data. Originally introduced in another form in [58], ϵ vanishes in the SU(3) chiral limit, a result known as Dashen's theorem. However, in the 1990's numerous phenomenological papers pointed out that ϵ might be an $\mathcal{O}(1)$ number, indicating a significant failure of SU(3) χPT in the description of electromagnetic effects on light-meson masses. However, the phenomenological determinations of ϵ feature some level of controversy, leading to the rather imprecise estimate $\epsilon = 0.7(5)$ given in the first edition of FLAG. Starting with the FLAG 19 edition of the review, we quote more precise averages for ϵ, directly obtained from lattice-QCD+QED simulations. We refer the reader to earlier editions of FLAG and to the review [59] for discussions of the phenomenological determinations of ϵ.

The quality criteria regarding finite-volume effects for calculations including QED are presented in Sec. 2.1.1. Due to the long-distance nature of the electromagnetic interaction, these effects are dominated by a power law in the lattice spatial size. The coefficients of this expansion depend on the chosen finite-volume formulation of QED. For $\mathrm{QED_L}$, these effects on the squared mass M^2 of a charged meson are given by [60–62]

$$\Delta_{\rm FV} M^2 = \alpha M^2 \left\{ \frac{c_1}{ML} + \frac{2c_1}{(ML)^2} + \mathcal{O}\left[\frac{1}{(ML)^3}\right] \right\}, \qquad (40)$$

with $c_1 \simeq -2.83730$. It has been shown in [60] that the two first orders in this expansion are exactly known for hadrons, and are equal to the pointlike case. However, the $\mathcal{O}[1/(ML)^3]$ term and higher orders depend on the structure of the hadron. The universal corrections for $\mathrm{QED_{TL}}$ can also be found in [60]. In all this part, for all computations using such universal formulae, the QED finite-volume quality criterion has been applied with $n_{\rm min} = 3$, otherwise $n_{\rm min} = 1$ was used (see 2.1.1).

Since FLAG 21, one new result has been reported for nondegenerate light-quark masses, namely CLQCD 23 [10]. This result is based on a new set of $N_f = 2+1$ stout-smeared clover fermion simulations, including one ensemble at the physical light-quark mass. This calculation achieves a ★ rating in all criteria except the inclusion of isospin-breaking effects. Regarding the latter, $(\Delta M_K^2)^\gamma$ from RM123 17 [63] is used to estimate the QCD kaon-mass splitting required to constrain m_u and m_d. Because of the use of a result already averaged for $N_f = 2+1+1$ up- and down-quark masses, and in application of our quality criterion, we do not include CLQCD 23 in our average for m_u/m_d.

Regarding results already presented in previous FLAG editions, we start by reviewing predictions for the $N_f = 2+1$ sector. MILC 09A [25] uses the mass difference between K^0 and K^+, from which they subtract electromagnetic effects using Dashen's theorem with corrections, as discussed in the introduction of this section. The up and down sea quarks remain degenerate in their calculation, fixed to the value of m_{ud} obtained from M_{π^0}. To determine m_u/m_d, BMW 10A, 10B [16, 17] follow a slightly different strategy. They obtain this ratio from their result for m_s/m_{ud} combined with a phenomenological determination of the isospin-breaking quark-mass ratio $Q = 22.3(8)$, from $\eta \to 3\pi$ decays [69] (the decay $\eta \to 3\pi$ is very sensitive to QCD isospin breaking, but fairly insensitive to QED isospin breaking). Instead of subtracting electromagnetic effects using phenomenology, RBC 07 [70] and Blum 10 [22] actually include a quenched electromagnetic field in their calculation. This means that their results include corrections to Dashen's theorem, albeit only in the presence of quenched electromagnetism. Since the up and down quarks in the sea are treated as degenerate, very small isospin corrections are neglected, as in MILC's calculation. PACS-CS 12 [14] takes the inclusion of isospin-breaking effects one step further. Using reweighting techniques, it also includes electromagnetic and $m_u - m_d$ effects in the sea. However, they do not correct for the large finite-volume effects coming from electromagnetism in their $M_\pi L \sim 2$ simulations, but provide rough estimates for their size, based on Ref. [71]. QCDSF/UKQCD 15 [67] uses QCD+QED dynamical simulations performed at the SU(3)-flavour-symmetric point, but at a single lattice spacing, so they do not enter our average. The smallest partially quenched ($m_{\rm sea} \neq m_{\rm val}$) pion mass is greater than 200 MeV, so our chiral-extrapolation criteria require a ○ rating. Concerning finite-volume effects, this work uses three spatial extents L of 1.6 fm, 2.2 fm, and 3.3 fm. QCDSF/UKQCD 15 claims that the volume dependence is not visible on the two largest volumes, leading them to assume that finite-size effects are under control. As a consequence of that, the final result for quark masses does not feature a finite-volume extrapolation or an estimation of the finite-volume uncertainty. However, in their work on the QED corrections to the hadron spectrum [67] based on the same ensembles, a volume study shows some level of compatibility with the $\mathrm{QED_L}$ finite-volume effects

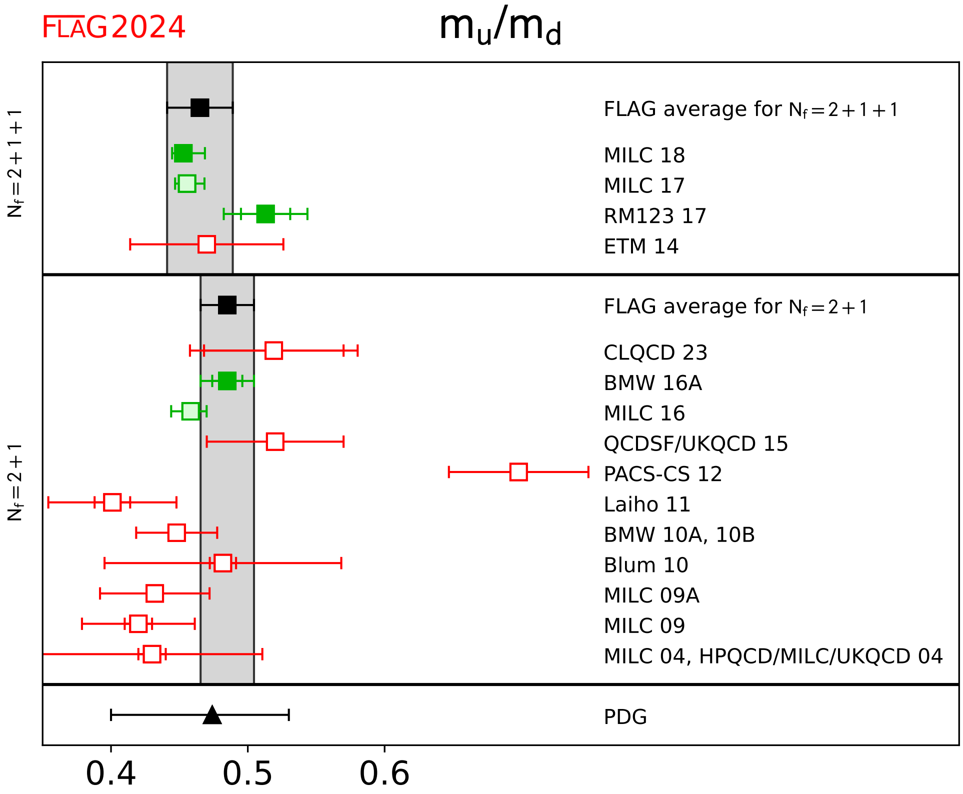

Figure 4: Lattice results and FLAG averages at $N_f = 2+1$ and $2+1+1$ for the up-down-quark masses ratio m_u/m_d, together with the current PDG estimate.

derived in [61]. We see two issues here. First, the analytical result quoted from [61] predicts large, $\mathcal{O}(10\%)$ finite-size effects from QED on the meson masses at the values of $M_\pi L$ considered in QCDSF/UKQCD 15, which is inconsistent with the statement made in the paper. Second, it is not known that the zero-mode regularization scheme used here has the same volume scaling as $\mathrm{QED_L}$. We therefore chose to assign the ■ rating for finite volume to QCDSF/UKQCD 15. BMW 16A [65] reuses the data set produced from their determination of the light-baryon octet-mass splittings [72] using electro-quenched $\mathrm{QCD{+}QED_{TL}}$ smeared clover-fermion simulations. Finally, MILC 16 [66], which is a preliminary result for the value of ϵ published in MILC 18 [64], also provides a $N_f = 2+1$ computation of the ratio m_u/m_d.

We now describe the $N_f = 2+1+1$ calculations. ETM 14 [41] uses simulations in pure QCD, but determines $m_u - m_d$ from the slope $\partial M_K^2/\partial m_{ud}$ and the physical value for the QCD kaon-mass splitting taken from the phenomenological estimate in FLAG 13. In the $N_f = 2+1+1$ sector, MILC 18 [64] computed ϵ using $N_f = 2+1$ asqtad electro-quenched $\mathrm{QCD{+}QED_{TL}}$ simulations and extracted the ratio m_u/m_d from a new set of $N_f = 2+1+1$ HISQ QCD simulations. Although ϵ comes from $N_f = 2+1$ simulations, $(\Delta M_K^2)^{\mathrm{SU(2)}}$, which is about three times larger than $(\Delta M_K^2)^\gamma$, has been determined in the $N_f = 2+1+1$ theory. We therefore chose to classify this result as a four-flavour one. This result is explicitly described by the authors as an update of MILC 17 [51]. In MILC 17 [51], m_u/m_d is determined as a side-product of a global analysis of heavy-meson decay constants, using a preliminary version of ϵ

from MILC 18 [64]. In FNAL/MILC/TUMQCD 18 [42] the ratio m_u/m_d from MILC 17 [51] is used to determine the individual masses m_u and m_d from a new calculation of m_{ud}. The work RM123 17 [63] is the continuation of the $N_f = 2$ work named RM123 13 [73] in the previous edition of FLAG. This group now uses $N_f = 2+1+1$ ensembles from ETM 10 [74], however, still with a rather large minimum pion mass of 270 MeV, leading to the ○ rating for chiral extrapolations.

Lattice results for m_u, m_d and m_u/m_d are summarized in Tab. 10. The colour coding is specified in detail in Sec. 2.1. Considering the important progress in the last years on including isospin-breaking effects in lattice simulations, we are now in a position where averages for m_u and m_d can be made without the need of phenomenological inputs. Therefore, lattice calculations of the individual quark masses using phenomenological inputs for isospin-breaking effects will be coded ■ .

We begin with $N_f = 2+1$ (for $N_f = 2$ see the 2021 edition). The only result that qualifies to enter the FLAG average is BMW 16A [65],

$$N_f = 2+1: \qquad \begin{aligned} m_u &= 2.27(9)\,\text{MeV} && \text{Ref. [65]}\,, \\ m_d &= 4.67(9)\,\text{MeV} && \text{Ref. [65]}\,, \\ m_u/m_d &= 0.485(19) && \text{Ref. [65]}\,, \end{aligned} \tag{41}$$

with errors of roughly 4%, 2% and 4%, respectively. These numbers result in the following RGI averages

$$N_f = 2+1: \qquad \begin{aligned} M_u^{\rm RGI} &= 3.15(12)_m(4)_\Lambda\,\text{MeV} && \text{Ref. [65]}\,, \\ M_d^{\rm RGI} &= 6.49(12)_m(7)_\Lambda\,\text{MeV} && \text{Ref. [65]}\,. \end{aligned} \tag{42}$$

Finally, for $N_f = 2+1+1$, RM123 17 [63] and FNAL/MILC/TUMQCD 18 [42] enter the average for the individual m_u and m_d masses, and RM123 17 [63] and MILC 18 [64] enter the average for the ratio m_u/m_d, giving

$$N_f = 2+1+1: \qquad \begin{aligned} m_u &= 2.14(8)\,\text{MeV} && \text{Refs. [42, 63]}\,, \\ m_d &= 4.70(5)\,\text{MeV} && \text{Refs. [42, 63]}\,, \\ m_u/m_d &= 0.465(24) && \text{Refs. [63, 64]}\,. \end{aligned} \tag{43}$$

with errors of roughly 4%, 1% and 5%, respectively. One can observe some marginal discrepancies between results coming from the MILC collaboration and RM123 17 [63]. More specifically, adding all sources of uncertainties in quadrature, one obtains a 1.7σ discrepancy between RM123 17 [63] and MILC 18 [64] for m_u/m_d, and a 2.2σ discrepancy between RM123 17 [63] and FNAL/MILC/TUMQCD 18 [42] for m_u. However, the values of m_d and ϵ are in very good agreement between the two groups. These discrepancies are presently too weak to constitute evidence for concern, and will be monitored as more lattice groups provide results for these quantities. The RGI averages for m_u and m_d are

$$N_f = 2+1+1: \qquad \begin{aligned} M_u^{\rm RGI} &= 2.97(11)_m(3)_\Lambda\,\text{MeV} && \text{Refs. [42, 63]}\,, \\ M_d^{\rm RGI} &= 6.53(7)_m(8)_\Lambda\,\text{MeV} && \text{Refs. [42, 63]}\,. \end{aligned} \tag{44}$$

Every result for m_u and m_d used here to produce the FLAG averages relies on electro-quenched calculations, so there is some interest to comment on the size of quenching effects.

Considering phenomenology and the lattice results presented here, it is reasonable for a rough estimate to use the value $(\Delta M_K^2)^\gamma \sim 2000$ MeV^2 for the QED part of the kaon-mass splitting. Using the arguments presented in Sec. B.1, one can assume that the QED sea contribution represents $\mathcal{O}(10\%)$ of $(\Delta M_K^2)^\gamma$. Using SU(3) PQχPT+QED [75, 76] gives a $\sim 5\%$ effect. Keeping the more conservative 10% estimate and using the experimental value of the kaon-mass splitting, one finds that the QCD kaon-mass splitting $(\Delta M_K^2)^{\mathrm{SU(2)}}$ suffers from a reduced 3% quenching uncertainty. Considering that this splitting is proportional to $m_u - m_d$ at leading order in SU(3) χPT, we can estimate that a similar error will propagate to the quark masses. So the individual up and down masses look mildly affected by QED quenching. However, one notices that $\sim 3\%$ is the level of error in the new FLAG averages, and increasing significantly this accuracy will require using fully dynamical calculations.

In view of the fact that a *massless up quark* would solve the strong CP problem, many authors have considered this an attractive possibility, but the results presented above exclude this possibility: the value of m_u in Eq. (41) differs from zero by 26 standard deviations. We conclude that nature solves the strong CP problem differently.

Finally, we conclude this section by giving the FLAG averages for ϵ defined in Eq. (38). For $N_f = 2+1+1$, we average the results of RM123 17 [63] and MILC 18 [64] with the value of $(\Delta M_K^2)^\gamma$ from BMW 14 [60] combined with Eq. (39), giving

$$N_f = 2+1+1: \qquad \epsilon = 0.79(6) \qquad \text{Refs. [60, 63, 64]}. \tag{45}$$

Although BMW 14 [60] focuses on hadron masses and did not extract the light-quark masses, they are the only fully unquenched QCD+QED calculation to date that qualifies to enter a FLAG average. With the exception of renormalization, which is not discussed in the paper, that work has a ★ rating for every FLAG criterion considered for the m_u and m_d quark masses. For $N_f = 2+1$ we use the results from BMW 16A [65],

$$N_f = 2+1: \qquad \epsilon = 0.73(17) \qquad \text{Ref. [65]}. \tag{46}$$

It is important to notice that the ϵ uncertainties from BMW 16A and RM123 17 are dominated by estimates of the QED quenching effects. Indeed, in contrast with the quark masses, ϵ is expected to be rather sensitive to the sea-quark QED contributions. Using the arguments presented in Sec. B.1, if one conservatively assumes that the QED sea contributions represent $\mathcal{O}(10\%)$ of $(\Delta M_K^2)^\gamma$, then Eq. (39) implies that ϵ will have a quenching error of ~ 0.15 for $(\Delta M_K^2)^\gamma \sim (45\ \mathrm{MeV})^2$, representing a large $\sim 20\%$ relative error. It is interesting to observe that such a discrepancy does not appear between BMW 14 and RM123 17, although the $\sim 10\%$ accuracy of both results might not be sufficient to resolve these effects. On the other hand, in the context of SU(3) chiral perturbation theory, Bijnens and Danielsson [75] show that the QED quenching effects on ϵ do not depend on unknown LECs at NLO in the chiral expansion and are therefore computable at that order. In that approach, MILC 18 finds the effect at NLO to be only 5%. To conclude, although the controversy around the value of ϵ has been significantly reduced by lattice-QCD+QED determinations, computing this at few-percent accuracy requires simulations with charged sea quarks.

4.1.4 Estimates for R and Q

The quark-mass ratios

$$R \equiv \frac{m_s - m_{ud}}{m_d - m_u} \quad \text{and} \quad Q^2 \equiv \frac{m_s^2 - m_{ud}^2}{m_d^2 - m_u^2} \tag{47}$$

compare SU(3) breaking with isospin breaking. Both numbers only depend on the ratios m_s/m_{ud} and m_u/m_d,

$$R = \frac{1}{2}\left(\frac{m_s}{m_{ud}} - 1\right)\frac{1+\frac{m_u}{m_d}}{1-\frac{m_u}{m_d}} \qquad \text{and} \qquad Q^2 = \frac{1}{2}\left(\frac{m_s}{m_{ud}} + 1\right) R\,. \tag{48}$$

The quantity Q is of particular interest because of a low-energy theorem [77], which relates it to a ratio of meson masses,

$$Q_M^2 \equiv \frac{\hat{M}_K^2}{\hat{M}_\pi^2}\frac{\hat{M}_K^2 - \hat{M}_\pi^2}{\hat{M}_{K^0}^2 - \hat{M}_{K^+}^2}\,, \qquad \hat{M}_\pi^2 \equiv \tfrac{1}{2}(\hat{M}_{\pi^+}^2 + \hat{M}_{\pi^0}^2)\,, \quad \hat{M}_K^2 \equiv \tfrac{1}{2}(\hat{M}_{K^+}^2 + \hat{M}_{K^0}^2)\,. \tag{49}$$

(We remind the reader that the $\hat{}$ denotes a quantity evaluated in the $\alpha \to 0$ limit.) Chiral symmetry implies that the expansion of Q_M^2 in powers of the quark masses (i) starts with Q^2 and (ii) does not receive any contributions at NLO [77]:

$$Q_M \overset{\text{NLO}}{=} Q\,. \tag{50}$$

For $N_f = 2+1$, we use Eqs. (36) and (41) and obtain

$$R = 38.1(1.5)\,, \qquad\qquad Q = 23.3(0.5)\,, \tag{51}$$

and for $N_f = 2+1+1$,

$$R = 35.9(1.7)\,, \qquad\qquad Q = 22.5(0.5)\,, \tag{52}$$

which are quite compatible (see the 2021 edition for the two flavour numbers which are also compatible with the above). It is interesting to note that the most recent phenomenological determination of R and Q from $\eta \to 3\pi$ decay [78] gives the values $R = 34.4(2.1)$ and $Q = 22.1(0.7)$, which are consistent with the averages presented here. The authors of Refs. [78, 79] point out that this discrepancy is likely due to surprisingly large corrections to the approximation in Eq. (50) used in the phenomenological analysis.

Our final results for the masses m_u, m_d, m_{ud}, m_s and the mass ratios m_u/m_d, m_s/m_{ud}, R, Q are collected in Tabs. 11 and 12.

Collaboration	Ref.	publication status	chiral extrapolation	continuum extrapolation	finite volume	renormalization	running	m_{ud}	m_s
CLQCD 23	[10]	A	★	★	★	★	e	3.60(11)(15)	98.8(2.9)(4.7)
ALPHA 19	[11]	A	○	★	★	★	e	3.54(12)(9)	95.7(2.5)(2.4)
Maezawa 16	[12]	A	■	★	★	★	d	–	92.0(1.7)
RBC/UKQCD 14B$^\ominus$	[13]	A	★	★	★	★	d	3.31(4)(4)	90.3(0.9)(1.0)
RBC/UKQCD 12$^\ominus$	[9]	A	★	○	★	★	d	3.37(9)(7)(1)(2)	92.3(1.9)(0.9)(0.4)(0.8)
PACS-CS 12*	[14]	A	★	■	■	★	b	3.12(24)(8)	83.60(0.58)(2.23)
Laiho 11	[15]	C	○	★	★	○	–	3.31(7)(20)(17)	94.2(1.4)(3.2)(4.7)
BMW 10A, 10B$^+$	[16, 17]	A	★	★	★	★	c	3.469(47)(48)	95.5(1.1)(1.5)
PACS-CS 10	[18]	A	★	■	■	★	b	2.78(27)	86.7(2.3)
MILC 10A	[19]	C	○	★	★	○	–	3.19(4)(5)(16)	–
HPQCD 10**	[20]	A	○	★	★	–	–	3.39(6)	92.2(1.3)
RBC/UKQCD 10A	[21]	A	○	○	★	★	a	3.59(13)(14)(8)	96.2(1.6)(0.2)(2.1)
Blum 10†	[22]	A	○	■	○	★	–	3.44(12)(22)	97.6(2.9)(5.5)
PACS-CS 09	[23]	A	★	■	■	★	b	2.97(28)(3)	92.75(58)(95)
HPQCD 09A$^\oplus$	[24]	A	○	★	★	–	–	3.40(7)	92.4(1.5)
MILC 09A	[25]	C	○	★	★	○	–	3.25 (1)(7)(16)(0)	89.0(0.2)(1.6)(4.5)(0.1)
MILC 09	[26]	A	○	★	★	○	–	3.2(0)(1)(2)(0)	88(0)(3)(4)(0)
PACS-CS 08	[27]	A	★	■	■	■	–	2.527(47)	72.72(78)
RBC/UKQCD 08	[28]	A	○	■	★	★	–	3.72(16)(33)(18)	107.3(4.4)(9.7)(4.9)
CP-PACS/ JLQCD 07	[29]	A	■	★	★	■	–	$3.55(19)(^{+56}_{-20})$	$90.1(4.3)(^{+16.7}_{-4.3})$
HPQCD 05	[30]	A	○	○	○	○	–	$3.2(0)(2)(2)(0)^\ddagger$	$87(0)(4)(4)(0)^\ddagger$
MILC 04, HPQCD/ MILC/UKQCD 04	[31, 32]	A	○	○	○	■	–	2.8(0)(1)(3)(0)	76(0)(3)(7)(0)

$^\ominus$ The results are given in the $\overline{\text{MS}}$ scheme at 3 instead of 2 GeV. We run them down to 2 GeV using numerically integrated 4-loop running [33, 34] with $N_f = 3$ and with the values of $\alpha_s(M_Z)$, m_b, and m_c taken from Ref. [35]. The running factor is 1.106. At three loops it is only 0.2% smaller, indicating that perturbative running uncertainties are small. We neglect them here.

* The calculation includes electromagnetic and $m_u \neq m_d$ effects through reweighting.

$^+$ The fermion action used is tree-level improved.

** m_s is obtained by combining m_c and HPQCD 09A's $m_c/m_s = 11.85(16)$ [24]. Finally, m_{ud} is determined from m_s with the MILC 09 result for m_s/m_{ud}. Since m_c/m_s is renormalization group invariant in QCD, the renormalization and running of the quark masses enter indirectly through that of m_c (see below).

† The calculation includes quenched electromagnetic effects.

$^\oplus$ What is calculated is $m_c/m_s = 11.85(16)$. m_s is then obtained by combining this result with the determination $m_c(m_c) = 1.268(9)$ GeV from Ref. [36]. Finally, m_{ud} is determined from m_s with the MILC 09 result for m_s/m_{ud}.

‡ The bare numbers are those of MILC 04. The masses are simply rescaled, using the ratio of the 2-loop to 1-loop renormalization factors.

a The masses are renormalized nonperturbatively at a scale of 2 GeV in a couple of $N_f = 3$ RI-SMOM schemes. A careful study of perturbative matching uncertainties has been performed by comparing results in the two schemes in the region of 2 GeV to 3 GeV [21].

b The masses are renormalized and run nonperturbatively up to a scale of 40 GeV in the $N_f = 3$ SF scheme. In this scheme, nonperturbative and NLO running for the quark masses are shown to agree well from 40 GeV all the way down to 3 GeV [18].

c The masses are renormalized and run nonperturbatively up to a scale of 4 GeV in the $N_f = 3$ RI-MOM scheme. In this scheme, nonperturbative and N^{3}LO running for the quark masses are shown to agree from 6 GeV down to 3 GeV to better than 1% [17].

d All required running is performed nonperturbatively.

e Running is performed nonperturbatively from 200 MeV to the electroweak scale $\sim$ 100 GeV.

Table 7: $N_f = 2 + 1$ lattice results for the masses m_{ud} and m_s (MeV).

Collaboration	Ref.	publication status	chiral extrapolation	continuum extrapolation	finite volume	renormalization	running	m_{ud}	m_s
ETM 21A	[40]	A	★	★	★	★	−	$3.636(66)(^{+60}_{-57})$	$98.7(2.4)(^{+4.0}_{-3.2})$
HPQCD 18†	[43]	A	★	★	★	★	−		94.49(96)
FNAL/MILC/TUMQCD 18	[42]	A	★	★	★	★	−	3.404(14)(21)	92.52(40)(56)
HPQCD 14A $^{\oplus}$	[44]	A	★	★	★	−	−		93.7(8)
ETM 14$^{\oplus}$	[41]	A	○	★	★	★	−	3.70(13)(11)	99.6(3.6)(2.3)

† Bare-quark masses are renormalized nonperturbatively in the RI-SMOM scheme at scales $\mu \sim$ 2–5 GeV for different lattice spacings and translated to the $\overline{\rm MS}$ scheme. Perturbative running is then used to run all results to a reference scale $\mu = 3$ GeV.

$^{\oplus}$ As explained in the text, m_s is obtained by combining the results $m_c(5\,\text{GeV}; N_f = 4) = 0.8905(56)$ GeV and $(m_c/m_s)(N_f = 4) = 11.652(65)$, determined on the same data set. A subsequent scale and scheme conversion, performed by the authors, leads to the value 93.6(8). In the table, we have converted this to $m_s(2\,\text{GeV}; N_f = 4)$, which makes a very small change.

Table 8: $N_f = 2 + 1 + 1$ lattice results for the masses m_{ud} and m_s (MeV).

Collaboration	Ref.	N_f	publication status	chiral extrapolation	continuum extrapolation	finite volume	m_s/m_{ud}
ETM 21A	[40]	2+1+1	A	★	★	★	$27.17(32)^{+56}_{-38}$
MILC 17 ‡	[51]	2+1+1	A	★	★	★	$27.178(47)^{+86}_{-57}$
FNAL/MILC 14A	[52]	2+1+1	A	★	★	★	$27.35(5)^{+10}_{-7}$
ETM 14	[41]	2+1+1	A	○	★	○	26.66(32)(2)
CLQCD 23	[10]	2+1	A	★	★	★	27.47(30)(13)
ALPHA 19	[37]	2+1	A	○	★	★	27.0(1.0)(0.4)
RBC/UKQCD 14B	[13]	2+1	A	★	★	★	27.34(21)
RBC/UKQCD 12⊖	[9]	2+1	A	★	○	★	27.36(39)(31)(22)
PACS-CS 12⋆	[14]	2+1	A	★	■	■	26.8(2.0)
Laiho 11	[15]	2+1	C	○	★	★	28.4(0.5)(1.3)
BMW 10A, 10B+	[16, 17]	2+1	A	★	★	★	27.53(20)(8)
RBC/UKQCD 10A	[21]	2+1	A	○	○	★	26.8(0.8)(1.1)
Blum 10†	[22]	2+1	A	○	■	○	28.31(0.29)(1.77)
PACS-CS 09	[23]	2+1	A	★	■	■	31.2(2.7)
MILC 09A	[25]	2+1	C	○	★	★	27.41(5)(22)(0)(4)
MILC 09	[26]	2+1	A	○	★	★	27.2(1)(3)(0)(0)
PACS-CS 08	[27]	2+1	A	★	■	■	28.8(4)
RBC/UKQCD 08	[28]	2+1	A	○	■	★	28.8(0.4)(1.6)
MILC 04, HPQCD/ MILC/UKQCD 04	[31, 32]	2+1	A	○	○	○	27.4(1)(4)(0)(1)

‡ The calculation includes electromagnetic effects.
⊖ The errors are statistical, chiral and finite volume.
⋆ The calculation includes electromagnetic and $m_u \neq m_d$ effects through reweighting.
+ The fermion action used is tree-level improved.
† The calculation includes quenched electromagnetic effects.

Table 9: Lattice results for the ratio m_s/m_{ud}.

Collaboration	Ref.	publication status	chiral extrapolation	continuum extrapolation	finite volume	isospin breaking	renormalization	running	m_u	m_d	m_u/m_d
MILC 18	[64]	A	★	★	★	○	★	–			$0.4529(48)(^{+150}_{-67})$
FNAL/MILC/TUMQCD 18*	[42]	A	★	★	★	○	★	–	2.118(17)(32)(12)(03)	4.690(30)(36)(26)(06)	
MILC 17†	[51]	A	★	★	★	○	★	–			$0.4556(55)(^{+114}_{-67})(13)$
RM123 17	[63]	A	○	★	★	○	★	*b*	2.50(15)(8)(2)	4.88(18)(8)(2)	0.513(18)(24)(6)
ETM 14	[41]	A	★	★	★	■	★	*b*	2.36(24)	5.03(26)	0.470(56)
CLQCD 23	[10]	A	★	★	★	■	★	*c*	2.45(22)(20)	4.74(11)(09)	0.519(51)(34)
BMW 16A	[65]	A	★	★	★	○	★	–	2.27(6)(5)(4)	4.67(6)(5)(4)	0.485(11)(8)(14)
MILC 16	[66]	C	○	★	★	○	★	–			$0.4582(38)(^{+12}_{-82})(1)(110)$
QCDSF/UKQCD 15	[67]	A	○	■	■	★	–	–			0.52(5)
PACS-CS 12	[14]	A	★	■	■	★	★	*a*	2.57(26)(7)	3.68(29)(10)	0.698(51)
Laiho 11	[15]	C	○	★	★	■	○	–	1.90(8)(21)(10)	4.73(9)(27)(24)	0.401(13)(45)
HPQCD 10‡	[20]	A	○	★	★	■	★	–	2.01(14)	4.77(15)	
BMW 10A, 10B+	[16, 17]	A	★	★	★	■	★	*b*	2.15(03)(10)	4.79(07)(12)	0.448(06)(29)
Blum 10	[22]	A	○	■	○	○	★	–	2.24(10)(34)	4.65(15)(32)	0.4818(96)(860)
MILC 09A	[25]	C	○	★	★	■	○	–	1.96(0)(6)(10)(12)	4.53(1)(8)(23)(12)	0.432(1)(9)(0)(39)
MILC 09	[26]	A	○	★	★	■	○	–	1.9(0)(1)(1)(1)	4.6(0)(2)(2)(1)	0.42(0)(1)(0)(4)
MILC 04, HPQCD/ MILC/UKQCD 04	[31][32]	A	○	○	○	■	■	–	1.7(0)(1)(2)(2)	3.9(0)(1)(4)(2)	0.43(0)(1)(0)(8)

* FNAL/MILC/TUMQCD 18 uses ϵ from MILC 18 to produce the individual m_u and m_d masses.

† MILC 17 additionally quotes an optional 0.0032 uncertainty on m_u/m_d corresponding to QED and QCD separation scheme ambiguities. Because this variation is not per se an error on the determination of m_u/m_d, and because it is generally not included in other results, we choose to omit it here.

‡ Values obtained by combining the HPQCD 10 result for m_s with the MILC 09 results for m_s/m_{ud} and m_u/m_d.

+ The fermion action used is tree-level improved.

a The masses are renormalized and run nonperturbatively up to a scale of 100 GeV in the $N_f = 2$ SF scheme. In this scheme, nonperturbative and NLO running for the quark masses are shown to agree well from 100 GeV all the way down to 2 GeV [68].

b The masses are renormalized and run nonperturbatively up to a scale of 4 GeV in the $N_f = 3$ RI-MOM scheme. In this scheme, nonperturbative and N^{3}LO running for the quark masses are shown to agree from 6 GeV down to 3 GeV to better than 1% [17].

c The masses are renormalized and run nonperturbatively in both the RI/MOM and SMOM schemes. The quoted quark-mass value is the RI/MOM one, with an assigned systematic error coming from the difference between the two schemes.

Table 10: Lattice results for m_u, m_d (MeV) and for the ratio m_u/m_d. The values refer to the $\overline{\rm MS}$ scheme at scale 2 GeV. The top part of the table lists the results obtained with $N_f = 2+1+1$, while the lower part presents calculations with $N_f = 2+1$.

N_f	m_{ud}	m_s	m_s/m_{ud}
2+1+1	3.410(43)	93.44(68)	27.23(10)
2+1	3.364(41)	92.03(88)	27.42(12)

Table 11: Our estimates for the average up-down-quark mass and the strange-quark mass in the $\overline{\text{MS}}$ scheme at running scale $\mu = 2\,\text{GeV}$. Mass values are given in MeV. In the results presented here, the error is the one which we obtain by applying the averaging procedure of Sec. 2.3 to the relevant lattice results.

N_f	m_u	m_d	m_u/m_d	R	Q
2+1+1	2.14(8)	4.70(5)	0.465(24)	35.9(1.7)	22.5(0.5)
2+1	2.27(9)	4.67(9)	0.485(19)	38.1(1.5)	23.3(0.5)

Table 12: Our estimates for the masses of the two lightest quarks and related, strong isospin-breaking ratios. Again, the masses refer to the $\overline{\text{MS}}$ scheme at running scale $\mu = 2\,\text{GeV}$. Mass values are given in MeV.

4.2 Charm-quark mass

In the following, we collect and discuss the lattice determinations of the $\overline{\rm MS}$ charm-quark mass $\overline{m}_c$. Most of the results have been obtained by analyzing the lattice-QCD simulations of two-point heavy-light- or heavy-heavy-meson correlation functions, using as input the experimental values of the D, D_s, and charmonium mesons. Some groups use the moments method. The latter is based on the lattice calculation of the Euclidean time moments of pseudoscalar-pseudoscalar correlators for heavy-quark currents followed by an OPE expansion dominated by perturbative QCD effects, which provides the determination of both the heavy-quark mass and the strong-coupling constant α_s.

The heavy-quark actions adopted by various lattice collaborations have been discussed in previous FLAG reviews [7, 53, 80], and their descriptions can be found in Sec. A.1.3 of FLAG 19 [53]. While the charm mass determined with the moments method does not need any lattice evaluation of the mass-renormalization constant Z_m, the extraction of $\overline{m}_c$ from two-point heavy-meson correlators does require the nonperturbative calculation of Z_m. The lattice scale at which Z_m is obtained is usually at least of the order 2–3 GeV, and therefore it is natural in this review to provide the values of $\overline{m}_c(\mu)$ at the renormalization scale $\mu = 3$ GeV. Since the choice of a renormalization scale equal to $\overline{m}_c$ is still commonly adopted (as by the PDG [1]), we have collected in Tab. 13 the lattice results for both $\overline{m}_c(\overline{m}_c)$ and $\overline{m}_c(3\text{ GeV})$, obtained for $N_f = 2+1$ and $2+1+1$. For $N_f = 2$, interested readers are referred to previous reviews [7, 80].

When not directly available in the published work, we apply a conversion factor using perturbative QCD evolution at five loops to run down from $\mu = 3$ GeV to the scales $\mu = \overline{m}_c$ and 2 GeV of 0.7739(60) and 0.9026(23), respectively, where the error comes from the uncertainty in $\Lambda_{\rm QCD}$. We use $\Lambda_{\rm QCD} = 297(12)$ MeV for $N_f = 4$ (see Sec. 9). Perturbation theory uncertainties, estimated as the difference between results that use 4- and 5-loop running, are significantly smaller than the parametric uncertainty coming from $\Lambda_{\rm QCD}$. For $\mu = \overline{m}_c$, the former is about about 2.5 times smaller.

In the next subsections, we review separately the results for $\overline{m}_c$ with three or four flavours of quarks in the sea.

4.2.1 $N_f = 2+1$ results

Since the last review [88], there is one new result: ALPHA 23 [83]. This work uses a subset of CLS ensembles, based on simulations of nonperturbatively $O(a)$-improved Wilson fermions. The difference with ALPHA 21 is that the valence sector uses both Wilson and twisted-mass discretizations instead of just Wilson. Renormalization is based on previous work by the ALPHA collaboration, and is performed nonperturbatively from 100 MeV to the electroweak scale. The subset of ensembles used have large volumes, four lattice spacings, and reach pion masses of 200 MeV, which guarantees entering in the average. Contrary to the extraction of light-quark masses in ALPHA 19, the chiral extrapolation does not dominate the error budget, and being less critical in this case we decide to give a ★ for the chiral extrapolation. The data-driven criteria quantity for the continuum extrapolation $\delta(a_{\rm min})$ (see 2.1.2) is smaller than 3 in all cases.

Petreczky 19 employs the HISQ action on ten ensembles with ten lattice spacings down to 0.025 fm, physical strange-quark mass, and two light-quark masses, the lightest corresponding to 161 MeV pions. Their study incorporates lattices with 11 different sizes, ranging from 1.6

Collaboration	Ref.	N_f	publication status	chiral extrapolation	continuum extrapolation	finite volume	renormalization	$\overline{m}_c(\overline{m}_c)$	$\overline{m}_c(3\text{ GeV})$
ETM 21A	[40]	2+1+1	P	★	★	★	★	$1.339(22)(^{+19}_{-10})(10)^\dagger$	$1.036(17)(^{+15}_{-8})$
HPQCD 20A	[81]	2+1+1	A	★	★	★	★	1.2719(78)	0.9841(51)
HPQCD 18	[43]	2+1+1	A	★	★	★	★	1.2757(84)	0.9896(61)
FNAL/MILC/ TUMQCD 18	[42]	2+1+1	A	★	★	★	–	1.273(4)(1)(10)	0.9837(43)(14)(33)(5)
HPQCD 14A	[44]	2+1+1	A	★	★	★	–	1.2715(95)	0.9851(63)
ETM 14A	[82]	2+1+1	A	○	★	○	★	1.3478(27)(195)	$1.0557(22)(153)^*$
ETM 14	[41]	2+1+1	A	○	★	○	★	1.348(46)	$1.058(35)^*$
ALPHA 23	[83]	2+1	A^+	★	★	★	★	1.296(15)	1.006(9)
ALPHA 21	[84]	2+1	A^+	★	★	★	★	1.296(19)	1.007(16)
Petreczky 19	[85]	2+1	A	★	★	★	★	1.265(10)	1.001(16)
Maezawa 16	[12]	2+1	A	■	★	★	★	1.267(12)	
JLQCD 16	[86]	2+1	A	○	★	★	–	1.2871(123)	1.0033(96)
χQCD 14	[87]	2+1	A	○	○	○	★	1.304(5)(20)	1.006(5)(22)
HPQCD 10	[20]	2+1	A	○	★	○	–	1.273(6)	0.986(6)
HPQCD 08B	[36]	2+1	A	○	★	○	–	1.268(9)	0.986(10)
PDG	[1]							1.27(2)	

† We applied the running factor 0.7739(60) for $\mu = 3$ GeV to $\overline{m}_c$. The errors are statistical, systematic, and the uncertainty in the running factor.

* A running factor equal to 0.900 between the scales $\mu = 2$ GeV and $\mu = 3$ GeV was applied by us.

$^+$ Published after the FLAG deadline.

Table 13: Lattice results for the $\overline{\text{MS}}$ charm-quark mass $\overline{m}_c(\overline{m}_c)$ and $\overline{m}_c(3\text{ GeV})$ in GeV, together with the colour coding of the calculations used to obtain them.

to 5.4 fm. The masses are computed from moments of pseudoscalar quarkonium correlation functions, and $\overline{\text{MS}}$ masses are extracted with 4-loop continuum perturbation theory. Thus, that work easily rates green stars in all categories. Continuum extrapolations are challenging, but judging the data itself the values of $\delta(a_{\text{min}})$ are not very large. It is just that the functional form of the data is complicated.

ALPHA 21 uses the $\mathcal{O}(a)$-improved Wilson-clover action with five lattice spacings from 0.087 to 0.039 fm, produced by the CLS collaboration. For each lattice spacing, several light sea-quark masses are used in a global chiral-continuum extrapolation (the lightest pion mass for one ensemble is 198 MeV). The authors also use nonperturbative renormalization and running through application of step-scaling and the Schrödinger functional scheme. Finite-volume effects are investigated at one lattice spacing and only for $\sim$ 400 MeV pions on the smallest two volumes where results are compatible within statistical errors. ALPHA 21 satisfies the FLAG criteria for green-star ratings in all of the categories listed in Tab. 13. The values of $\delta(a_{\text{min}})$ are smaller than 3 in all continuum extrapolations. Descriptions of the other works in this section can be found in an earlier review [53].

According to our rules on the publication status, the FLAG average for the charm-quark mass at $N_f = 2+1$ is obtained by combining the results HPQCD 10, χQCD 14, JLQCD 16, Petreczky 19, ALPHA 21 and ALPHA 23,

$$N_f = 2+1: \qquad \overline{m}_c(\overline{m}_c) = 1.278(6) \text{ GeV} \qquad \text{Refs. [20, 83–87]}, \tag{53}$$

$$\overline{m}_c(3 \text{ GeV}) = 0.991(6) \text{ GeV} \qquad \text{Refs. [20, 83–87]}, \tag{54}$$

This result corresponds to the following RGI average

$$M_c^{\text{RGI}} = 1.526(7)_m(21)_\Lambda \text{ GeV} \qquad \text{Refs. [20, 84–87]}. \tag{55}$$

4.2.2 $N_f = 2+1+1$ results

For a discussion of older results, see the previous FLAG reviews. Since FLAG 19 two groups have produced updated values with charm quarks in the sea.

HPQCD 20A [81] is an update of HPQCD 18, including a new finer ensemble ($a \approx 0.045$ fm) and EM corrections computed in the quenched approximation of QED for the first time. Besides these new items, the analysis is largely unchanged from HPQCD 18 except for an added α_s^3 correction to the SMOM-to-$\overline{\text{MS}}$ conversion factor and tuning the bare charm mass via the J/ψ mass rather than the η_c. Their new value in pure QCD is $\overline{m}_c(3 \text{ GeV}) = 0.9858(51)$ GeV which is quite consistent with HPQCD 18 and the FLAG 19 average. The effects of quenched QED in both the bare charm-quark mass and the renormalization constant are small. Both effects are precisely determined, and the overall effect shifts the mass down slightly to $\overline{m}_c(3 \text{ GeV}) = 0.9841(51)$ where the uncertainty due to QED is invisible in the final error. The shift from their pure QCD value due to quenched QED is about -0.2%.

ETM 21A [40] is a new work that follows a similar methodology as ETM 14, but with significant improvements. Notably, a clover-term is added to the twisted mass fermion action which suppresses $\mathcal{O}(a^2)$ effects between the neutral and charged pions. Additional improvements include new ensembles lying very close to the physical mass point, better control of nonperturbative renormalization systematics, and use of both meson and baryon correlation functions to determine the quark mass. They use the RI-MOM scheme for nonperturbative renormalization. The analysis comprises ten ensembles in total with three lattice spacings (0.095, 0.082, and 0.069 fm), two volumes for the finest lattice spacings and four for the other two, and pion masses down to 134 MeV for the finest ensemble. The values of $m_\pi L$ range mostly from almost four to greater than five. According to the FLAG criteria, green stars are earned in all categories. The authors find $m_c(3 \text{ GeV}) = 1.036(17)(^{+15}_{-8})$ GeV. In Tab. 13 we have applied a factor of 0.7739(60) to run from 3 GeV to $\overline{m}_c$. As in FLAG 19, the new value is consistent with ETM 14 and ETM 14A, but is still high compared to the FLAG average. The authors plan future improvements, including a finer lattice spacing for better control of the continuum limit and a new renormalization scheme, like RI-SMOM.

Six results enter the FLAG average for $N_f = 2+1+1$ quark flavours: ETM 14, ETM 14A, HPQCD 14A, FNAL/MILC/TUMQCD 18, HPQCD 20A, and ETM 21A. We note that while the ETM determinations of $\overline{m}_c$ agree well with each other, they are incompatible with HPQCD 14A, FNAL/MILC/TUMQCD 18, and HPQCD 20A by several standard deviations. While the ETM 14 and ETM 14A use the same configurations, the analyses are quite different and independent, and ETM 21A is a new result on new ensembles with improved methodology.

As mentioned earlier, m_{ud} and m_s values by ETM are also systematically high compared to their respective averages. Combining all six results yields yields

$$N_f = 2+1+1: \qquad \overline{m}_c(\overline{m}_c) = 1.280(13) \text{ GeV} \qquad \text{Refs. [40–42, 44, 81, 82]}, \qquad (56)$$
$$\overline{m}_c(3 \text{ GeV}) = 0.989(10) \text{ GeV} \qquad \text{Refs. [40–42, 44, 81, 82]}, \qquad (57)$$

where the errors include large stretching factors $\sqrt{\chi^2/\text{dof}} \approx 2.0$ and 2.4, respectively. We have assumed 100% correlation for statistical errors between ETM 14 and ETM 14A results and the same for HPQCD 14A, HPQCD 20A, and FNAL/MILC/TUMQCD 18.

These are obviously poor χ^2 values, and the stretching factors are quite large. While it may be prudent in such a case to quote a range of values covering the central values of all results that pass the quality criteria, we believe in this case that would obscure rather than clarify the situation. From Fig. 5, we note that not only do ETM 21A, ETM 14A, and ETM 14 lie well above the other 2+1+1 results, but also above all of the 2+1 flavour results. A similar trend is apparent for the light-quark masses (see Figs. 1 and 2) while for mass ratios there is better agreement (Figs. 3, 4, and 6). The latter suggests there may be underestimated systematic uncertainties associated with scale setting and/or renormalization which have not been detected. Finally we note the ETM results are significantly higher than the PDG average. For these reasons, which admittedly are not entirely satisfactory, we continue to quote an average with a stretching factor as in previous reviews.

The RGI average reads as follows,

$$M_c^{\text{RGI}} = 1.528(15)_m(21)_\Lambda \text{ GeV} \qquad \text{Refs. [40–42, 44, 81, 82]}. \qquad (58)$$

Figure 5 presents the values of $\overline{m}_c(\overline{m}_c)$ given in Tab. 13 along with the FLAG averages obtained for $2+1$ and $2+1+1$ flavours.

4.2.3 Lattice determinations of the ratio m_c/m_s

Because some of the results for quark masses given in this review are obtained via the quark-mass ratio m_c/m_s, we review these lattice calculations, which are listed in Tab. 14, as well.

The $N_f = 2+1$ results from χQCD 14 and HPQCD 09A [24] are from the same calculations that were described for the charm-quark mass in the previous review. Maezawa 16 does not pass our chiral-limit test (see the previous review), though we note that it is quite consistent with the other values. Combining χQCD 14 and HPQCD 09A, we obtain the same result reported in FLAG 19,

$$N_f = 2+1: \qquad m_c/m_s = 11.82(16) \qquad \text{Refs. [24, 87]}, \qquad (59)$$

with a $\chi^2/\text{dof} \simeq 0.85$.

Turning to $N_f = 2+1+1$, there is a new result from ETM 21A (see the previous section for details). The errors have actually increased compared to ETM 14, due to larger uncertainties in the baryon sector which enter their average with the meson sector. See the earlier reviews for a discussion of previous results.

We note that some tension exists between the HPQCD 14A and FNAL/MILC/TUMQCD results. Combining these with ETM 14 and ETM 21A yields

$$N_f = 2+1+1: \qquad m_c/m_s = 11.766(30) \qquad \text{Refs. [40–42, 44]}, \qquad (60)$$

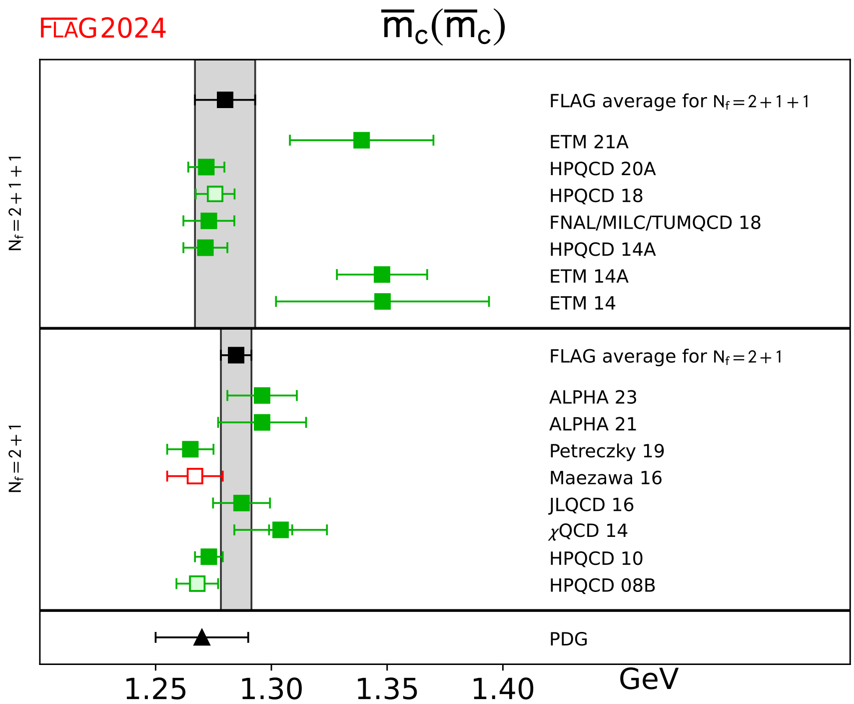

Figure 5: The charm-quark mass for $2+1$ and $2+1+1$ flavours. For the latter a large stretching factor is used for the FLAG average due to poor χ^2 from our fit.

Collaboration	Ref.	N_f	publication status	chiral extrapolation	continuum extrapolation	finite volume	m_c/m_s
ETM 21A	[40]	2+1+1	P	★	★	★	$11.48(12)(^{+25}_{-19})$
FNAL/MILC/TUMQCD 18	[42]	2+1+1	A	★	★	★	11.784(11)(17)(00)(08)
HPQCD 14A	[44]	2+1+1	A	★	★	★	11.652(35)(55)
ETM 14	[41]	2+1+1	A	○	★	○	11.62(16)
Maezawa 16	[12]	2+1	A	■	★	★	11.877(91)
χQCD 14	[87]	2+1	A	○	○	○	11.1(8)
HPQCD 09A	[24]	2+1	A	○	★	★	11.85(16)

Table 14: Lattice results for the quark-mass ratio m_c/m_s, together with the colour coding of the calculations used to obtain them.

where the error includes the stretching factor $\sqrt{\chi^2/\text{dof}} \simeq 1.4$. We have assumed a 100% correlation of statistical errors for FNAL/MILC/TUMQCD 18 and HPQCD 14A.

Results for m_c/m_s are shown in Fig. 6 together with the FLAG averages for $N_f = 2+1$

and $2+1+1$ flavours.

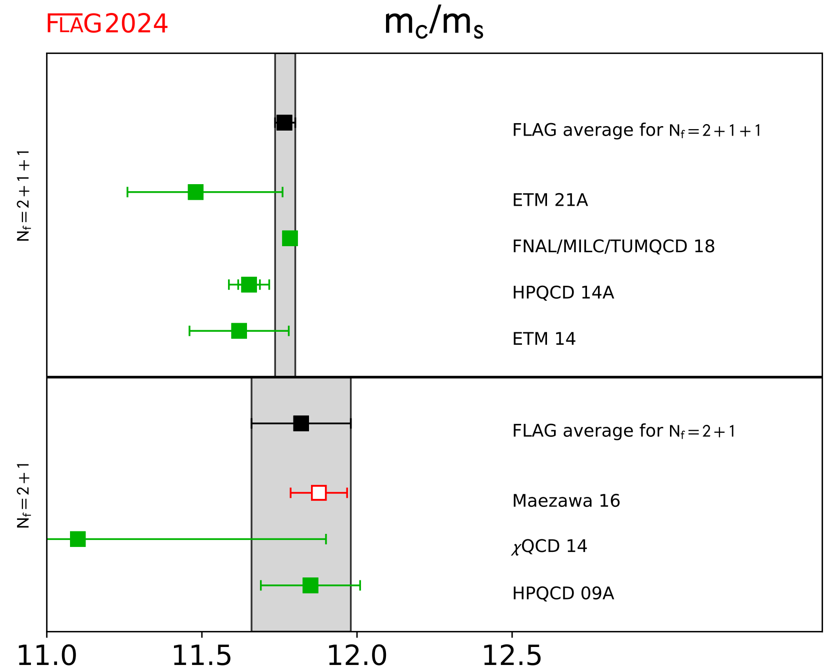

Figure 6: Lattice results for the ratio m_c/m_s listed in Tab. 14 and the FLAG averages corresponding to $2+1$ and $2+1+1$ quark flavours. The latter average includes a stretching factor of 1.4 on the error due a poor χ^2 from our fit.

4.3 Bottom-quark mass

Now we review the lattice results for the $\overline{\rm MS}$ bottom-quark mass $\overline{m}_b$. Related heavy-quark actions and observables have been discussed in previous FLAG reviews [7, 53, 80], and descriptions can be found in Sec. A.1.3 of FLAG 19 [53]. In Tab. 15, we collect results for $\overline{m}_b(\overline{m}_b)$ obtained with $N_f = 2+1$ and $2+1+1$ sea-quark flavours. Available results for the quark-mass ratio m_b/m_c are also reported. After discussing the new results, we evaluate the corresponding FLAG averages.

Collaboration	Ref.	N_f	publication status	chiral extrapolation	continuum extrapolation	finite volume	renormalization	heavy-quark treatment	$\overline{m}_b(\overline{m}_b)$	m_b/m_c
HPQCD 21	[89]	2+1+1	A	★	○	★	–	✓	$4.209(21)^{++}$	$4.586(12)^{**}$
FNAL/MILC/TUM 18	[42]	2+1+1	A	★	○	★	–	✓	4.201(12)(1)(8)(1)	4.578(5)(6)(0)(1)
Gambino 17	[90]	2+1+1	A	○	★	○	★	✓	4.26(18)	
ETM 16B	[91]	2+1+1	A	○	★	○	★	✓	$4.26(3)(10)^{+}$	4.42(3)(8)
HPQCD 14B	[92]	2+1+1	A	★	★	★	★	✓	$4.196(0)(23)^{\dagger}$	
Petreczky19	[85]	2+1	A	★	★	★	★	✓	4.188(37)	4.586(43)
Maezawa 16	[12]	2+1	A	■	★	★	★	✓	4.184(89)	4.528(57)
HPQCD 13B	[93]	2+1	A	■	○	–	–	✓	4.166(43)	
HPQCD 10	[20]	2+1	A	★	★	★	–	✓	4.164(23)	4.51(4)
ETM 13B	[94]	2	A	○	★	○	★	✓	4.31(9)(8)	
ALPHA 13C	[95]	2	A	★	★	★	★	✓	4.21(11)	
ETM 11A	[96]	2	A	○	★	○	★	✓	4.29(14)	
PDG	[1]								$4.18^{+0.02}_{-0.03}$	

$^{++}$ We quote the four-flavour result. For $N_f = 5$, the value is 4.202(21).
** The ratio is quoted in the $\overline{\rm MS}$ scheme for $\mu = 3$ GeV because of the different charges of the bottom and charm quarks.
† Only two pion points are used for chiral extrapolation.

Table 15: Lattice results for the $\overline{\rm MS}$ bottom-quark mass $\overline{m}_b(\overline{m}_b)$ in GeV, together with the systematic error ratings for each. Available results for the quark-mass ratio m_b/m_c are also reported.

4.3.1 $N_f = 2+1$

There are no new results since the last review, so we simply quote the same average of HPQCD 10 and Petreczky 19 (both are reported for $N_f = 5$, so we simply quote the average for $N_f = 5$).

$$N_f = 2+1: \qquad \overline{m}_b(\overline{m}_b) = 4.171(20)\ \text{GeV} \qquad \text{Refs. [20, 85]}. \tag{61}$$

The corresponding (four-flavour) RGI average is

$$N_f = 2+1: \qquad M_b^{\rm RGI} = 6.888(33)_m(45)_\Lambda \text{ GeV} \qquad \text{Refs. [20, 85]}. \qquad (62)$$

4.3.2 $N_f = 2+1+1$

HPQCD 21 [89] is an update of HPQCD 14A (and replaces it in our average. See FLAG 19 for details.), including EM corrections for the first time for the b-quark mass. Four flavours of HISQ quarks are used on MILC ensembles with lattice spacings from about 0.09 to 0.03 fm. Ensembles with physical- and unphysical-mass sea-quarks are used. Quenched QED is used to obtain the dominant $\mathcal{O}(\alpha)$ effect. The ratio of bottom- to charm-quark masses is computed in a completely nonperturbative formulation, and the b-quark mass is extracted using the value of $\overline{m}_c(3 \text{ GeV})$ from HPQCD 20A. Since EM effects are included, the QED renormalization scale enters the ratio which is quoted for 3 GeV and $N_f = 4$. The total error on the new result is more than two times smaller than for HPQCD 14A, but is only slightly smaller compared to the NRQCD result reported in HPQCD 14B. The inclusion of QED shifts the ratio m_b/m_c up slightly from the pure QCD value by about one standard deviation, and the value of $\overline{m}_b(\overline{m}_b)$ is consistent, within errors, to the other pure QCD results entering our average. Therefore, we quote a single average. Cutoff effects are significant in that work, and are the dominant source of uncertainty in the ratio m_b/m_c. It is difficult to estimate the value of $\delta(a_{\rm min})$ from the data present in the publication, but the authors provided extra information about their analysis with the result that $\delta(a_{\rm min}) \approx 3$. Therefore, we do not inflate the errors of that computation. The work rates green stars for all FLAG criteria except for the continuum limit (see Tab. 15) where less than three ensembles at the physical b-quark mass were used in the $a \to 0$ extrapolation (in the previous FLAG review this was missed and is corrected here).

HPQCD 14B employs the NRQCD action [92] to treat the b quark. The b-quark mass is computed with the moments method, that is, from Euclidean-time moments of two-point, heavy-heavy-meson correlation functions (see also Sec. 9.8 for a description of the method). Due to the effective treatment of the heavy quark, continuum extrapolations are under control since five lattice spacings are employed, with the smallest about 0.09 fm, but the requirement that $am_b \ll 1$ is not relevant. Their final result is $\overline{m}_b(\mu = 4.18\,\text{GeV}) = 4.207(26)$ GeV, where the error is from adding systematic uncertainties in quadrature only (statistical errors are smaller than 0.1% and ignored). The errors arise from renormalization, perturbation theory, lattice spacing, and NRQCD systematics. The finite-volume uncertainty is not estimated, but at the lowest pion mass they have $m_\pi L \simeq 4$, which leads to the tag ★ . In this case, the continuum extrapolations seem mild, in part, thanks to the NRQCD action used to treat the b quark. The data-driven continuum-limit criterion $\delta(a_{\rm min}) < 3$, so no correction factor is necessary here.

The next four-flavour result (ETM 16B [91]) is from the ETM collaboration and updates their preliminary result appearing in a conference proceedings [97]. The calculation is performed on a set of ensembles generated with twisted-Wilson fermions with three lattice spacings in the range 0.06 to 0.09 fm and with pion masses in the range 210 to 440 MeV. The b-quark mass is determined from a ratio of heavy-light pseudoscalar meson masses designed to yield the quark pole mass in the static limit. The pole mass is related to the $\overline{\rm MS}$ mass through perturbation theory at N^{3}LO. The key idea is that by taking ratios of ratios, the b-quark mass is accessible through fits to heavy-light(strange)-meson correlation functions computed on the lattice in the range ~ 1–$2 \times m_c$ and the static limit, the latter being exactly 1. By simulating below $\overline{m}_b$, taking the continuum limit is easier. They find $\overline{m}_b(\overline{m}_b) = 4.26(3)(10)$ GeV, where the first error is statistical and the second systematic. The dominant errors come from setting the lattice scale and fit systematics.

Gambino *et al.* [90] use twisted-mass-fermion ensembles from the ETM collaboration and the ETM ratio method as in ETM 16B. Three values of the lattice spacing are used, ranging from 0.062 to 0.089 fm. Several volumes are also used. The light-quark masses produce pions with masses from 210 to 450 MeV. The main difference with ETM 16 is that the authors use the kinetic mass defined in the heavy-quark expansion (HQE) to extract the b-quark mass instead of the pole mass. They include an additional uncertainty stemming from the conversion between kinetic and $\overline{\rm MS}$ schemes which leads to a somewhat larger total uncertainty compared to ETM 16B.

The final b-quark mass result is FNAL/MILC/TUM 18 [42]. The mass is extracted from the same fit and analysis done for the charm quark mass. Note that relativistic HISQ valence masses reach the physical b mass on the two finest lattice spacings ($a = 0.042$ fm, 0.03 fm) with physical and $0.2 \times m_s$ light-quark masses, respectively. In lattice units, the heavy valence masses correspond to $aM^{\rm RGI} > 0.90$, making the continuum extrapolation challenging. The extrapolations have $\delta(a_{\rm min}) \approx 14$ (taking into account only the statistical error of the continuum extrapolation, which is a 40% of their total error budget). According to our policy (2.1.2) we increase the error for the average by a factor 3.5. Their results are also consistent with an analysis dropping the finest lattice spacing from the fit. Since the b-quark mass region is only reached with two lattice spacings, we rate this work with a green circle for the continuum extrapolation (the same as HPQCD 21). Note, however, that for other values of the quark masses they use up to five values of the lattice spacing (cf. their charm-quark mass determination) with small values of $\delta(a_{\rm min})$ in the continuum extrapolation. In summary, we judge that these large scaling violations affect mainly the determination of the b-quark mass.

All of the above results enter our average. We note that here the ETM 16B result is consistent with the average and a stretching factor on the error is not used.

$$N_f = 2+1+1: \qquad \overline{m}_b(\overline{m}_b) = 4.200(14) \text{ GeV} \qquad \text{Refs. [42, 89–92]}. \tag{63}$$

We have included a 100% correlation on the statistical errors of ETM 16B and Gambino 17, since the same ensembles are used in both. While FNAL/MILC/TUM 18 and HPQCD 21 also use the same MILC HISQ ensembles, the statistical error in the HPQCD 21 analysis is negligible, so we do not include a correlation between them. The average has $\chi^2/\text{dof} = 0.02$.

The above translates to the RGI average

$$N_f = 2+1+1: \qquad M_b^{\rm RGI} = 6.938(23)_m(45)_\Lambda \text{ GeV} \qquad \text{Refs. [42, 89–92]}. \tag{64}$$

Results for $\overline{m}_b(\overline{m}_b)$ are shown in Fig. 7 together with the FLAG averages corresponding to $N_f = 2+1$ and $2+1+1$ quark flavours.

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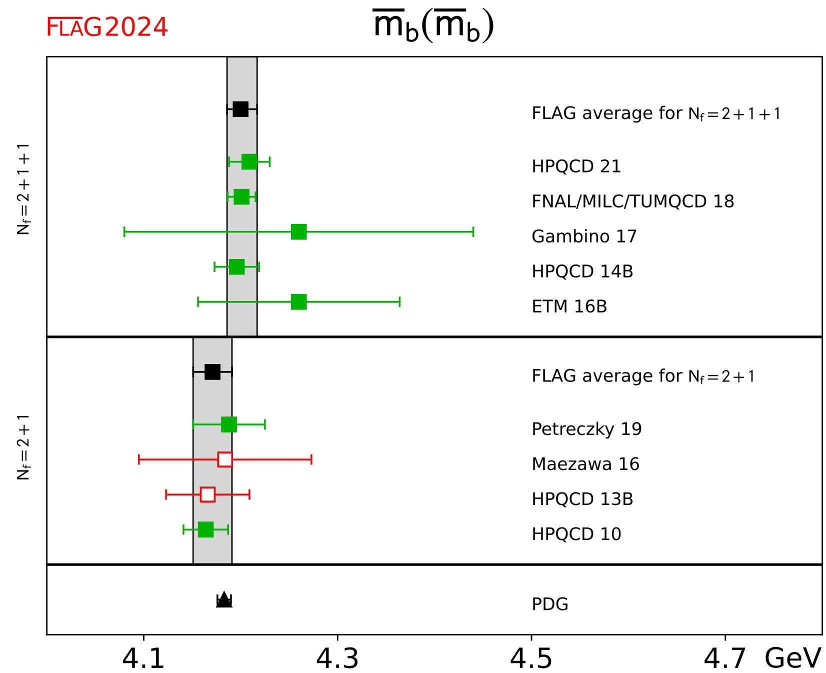

Figure 7: The b-quark mass for $N_f = 2+1$ and $2+1+1$ flavours. The updated PDG value from Ref. [98] is reported for comparison.

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